

# FIRST-PASSAGE PERCOLATION AND LOCAL MODIFICATIONS OF DISTANCES IN RANDOM TRIANGULATIONS\*

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## Abstract

We study local modifications of the graph distance in large random triangulations. Our main results show that, in large scales, the modified distance behaves like a deterministic constant  $\mathbf{c} \in (0, \infty)$  times the usual graph distance. This applies in particular to the first-passage percolation distance obtained by assigning independent random weights to the edges of the graph. We also consider the graph distance on the dual map, and the first-passage percolation on the dual map with exponential edge weights, which is closely related to the so-called Eden model. In the latter two cases, we are able to compute explicitly the constant  $\mathbf{c}$  by using earlier results about asymptotics for the peeling process. In general however, the constant  $\mathbf{c}$  is obtained from a subadditivity argument in the infinite half-plane model that describes the asymptotic shape of the triangulation near the boundary of a large ball. Our results apply in particular to the infinite random triangulation known as the UIPT, and show that balls of the UIPT for the modified distance are asymptotically close to balls for the graph distance.

## 1 Introduction

In the recent years, there has been much effort to understand the large-scale geometry of random planar maps viewed as random metric spaces for the usual graph distance on their vertex set. A major achievement in the area is the construction and study of the so-called Brownian map, which has been proved to be the universal scaling limit of many different classes of planar maps equipped with the graph distance (see [27, 33] and more recently [1, 2, 12]). An account of these developments can be found in the surveys [26, 30]. In the present work, we replace the graph distance by other natural choices of distances on the vertex set or on the set of faces, and we show that, in large scales, these new distances behave like the original graph distance, up to a constant multiplicative factor. In particular, we prove that the vertex set of a uniformly distributed random plane triangulation with  $n$  vertices equipped with the (suitably rescaled) first-passage percolation distance obtained by assigning independent random lengths to the edges converges in distribution to the Brownian map as  $n \rightarrow \infty$ , in the Gromov–Hausdorff sense. So, in some sense, the extra randomness coming from the weights assigned to the edges plays no role in the limit. This is a new illustration of the universality of the Brownian map as a two-dimensional model of random geometry. We mention here that the study of discrete or continuous models of random geometry has been strongly motivated by their relevance to various domains of theoretical physics, and in particular to the so-called two-dimensional quantum gravity. Discrete random geometry has been the

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subject of intensive research in the physics literature since the beginning of the eighties when Polyakov suggested to solve questions coming from string theory and quantum gravity by developing a formalism to calculate sums over random surfaces, as a kind of analog of the famous Feynman path integrals. We refer to the book [4] for an overview of the use of discrete random surfaces in theoretical physics.

Let us turn to a more detailed description of our main results. We recall that a planar map is a proper embedding of a finite connected (multi)graph in the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms of the sphere. We will always consider rooted planar maps, which means that there is a distinguished oriented edge whose initial vertex is called the root vertex. The faces are the connected components of the complement of edges, and a planar map is called a triangulation if all its faces are triangles (possibly with two sides glued together).

**Modified distances.** If  $m$  is a rooted planar map, we let  $V(m)$ ,  $E(m)$  and  $F(m)$  denote respectively the sets of vertices, edges and faces of  $m$ . The set  $V(m)$  is usually equipped with the graph distance, which is denoted by  $d_{\text{gr}}$  (or  $d_{\text{gr}}^m$  if there is a risk of ambiguity). We introduce the following modifications of the graph distance.

**Case 0. FIRST-PASSAGE (BOND) PERCOLATION.** Assign independent identically distributed positive random variables  $w(e)$  to all edges  $e \in E(m)$ . We assume that the common distribution of the “weights”  $w(e)$  is supported on  $[\kappa, 1]$  for some  $\kappa \in (0, 1]$ . The associated first-passage percolation distance is defined on  $V(m)$  by setting for any  $x, y \in V(m)$ ,

$$d_{\text{fpp}}(x, y) = \inf_{\gamma: x \rightarrow y} \sum_{e \in \gamma} w(e),$$

where the infimum runs over all paths  $\gamma$  going from  $x$  to  $y$  in the map  $m$ .

**Case 1. DUAL GRAPH DISTANCE.** Consider the dual map  $m^\dagger$ , whose vertices are the faces of  $m$ , and each edge  $e$  of  $m$  corresponds to an edge of  $m^\dagger$  connecting the two (possibly equal) faces incident to  $e$ . We may then consider the graph distance on  $V(m^\dagger) = F(m)$ , which we denote by  $d_{\text{gr}}^\dagger$ .

**Case 2. EDEN MODEL.** This is the first-passage percolation model on  $m^\dagger$  corresponding to exponential edge weights. More precisely, we assign independent exponential random variables with parameter 1 to the edges of  $m^\dagger$  (or equivalently to the edges of  $m$ ) and the associated first-passage percolation distance on  $F(m) = V(m^\dagger)$  is denoted by  $d_{\text{Eden}}^\dagger$ . We call this the Eden model because of the close relation with the classical Eden growth model, see in particular [3, Section 6] or [35, Section 1.2.1], and also [17, Proposition 15].

The functions  $d_{\text{gr}}$  and  $d_{\text{fpp}}$  are distances on  $V(m)$  whereas  $d_{\text{gr}}^\dagger$  and  $d_{\text{Eden}}^\dagger$  are distances on  $F(m)$ . To compare the latter distances to the usual graph metric on  $m$ , we will replace faces by incident vertices, and we use the notation  $x \triangleleft f$  to mean that the vertex  $x$  is incident to the face  $f$ .

**Finite triangulations.** We consider these “modified distances” when  $m = \mathcal{T}_n$  is a random planar map chosen uniformly at random in the set of all rooted plane triangulations with  $n + 1$  vertices (we consider type I triangulations where loops and multiple edges are allowed). In each of the previous cases, we are able to prove that the modified distances behave in large scales like a deterministic constant times the graph distance on  $V(\mathcal{T}_n)$ . More precisely, there exist constants  $\mathbf{c}_0, \mathbf{c}_1$  and  $\mathbf{c}_2$  in  $(0, \infty)$  such that we have

the following three convergences in probability

$$n^{-1/4} \sup_{x,y \in V(\mathcal{T}_n)} |d_{\text{fpp}}(x,y) - \mathbf{c}_0 \cdot d_{\text{gr}}(x,y)| \xrightarrow{n \rightarrow \infty} 0, \quad (1)$$

$$n^{-1/4} \sup_{\substack{x,y \in V(\mathcal{T}_n), f,g \in F(\mathcal{T}_n) \\ x \triangleleft f \text{ and } y \triangleleft g}} |d_{\text{gr}}^\dagger(f,g) - \mathbf{c}_1 \cdot d_{\text{gr}}(x,y)| \xrightarrow{n \rightarrow \infty} 0, \quad (2)$$

$$n^{-1/4} \sup_{\substack{x,y \in V(\mathcal{T}_n), f,g \in F(\mathcal{T}_n) \\ x \triangleleft f \text{ and } y \triangleleft g}} |d_{\text{Eden}}^\dagger(f,g) - \mathbf{c}_2 \cdot d_{\text{gr}}(x,y)| \xrightarrow{n \rightarrow \infty} 0. \quad (3)$$

Since the convergence of rescaled triangulations to the Brownian map [27] implies that the typical graph distance between two vertices of  $\mathcal{T}_n$  is of order  $n^{1/4}$ , the convergence (1) shows that in large scales  $d_{\text{fpp}}(x,y)$  is proportional to  $d_{\text{gr}}(x,y)$ . In fact (1) implies that the set  $V(\mathcal{T}_n)$  equipped with the metric  $n^{-1/4}d_{\text{fpp}}$  converges in distribution in the Gromov–Hausdorff sense to (a scaled version of) the Brownian map, and that this convergence takes place jointly with that of  $(V(\mathcal{T}_n), n^{-1/4}d_{\text{gr}})$  proved in [27] (see Corollary 23 below). Similarly (2) shows that uniform rooted trivalent maps with  $n$  faces (which are the dual maps of rooted triangulations with  $n$  vertices) converge after rescaling toward the Brownian map.

In case 0., the constant  $\mathbf{c}_0$  depends on the distribution of the weights and an explicit calculation of this constant seems hopeless. However in cases 1. and 2. (dual graph and Eden model) the constants can be computed exactly and we have

$$\mathbf{c}_1 = 1 + 2\sqrt{3} \quad \text{and} \quad \mathbf{c}_2 = 2\sqrt{3}.$$

The reason why these two models are more tractable is the fact that balls for the dual graph distance or for the Eden model can be generated and studied via an algorithmic procedure known as the peeling process, which was first discussed by Angel [6], following the work of Watabiki [40] in the physics literature. These peeling explorations for balls have been studied in detail in the case of the Uniform Infinite Planar Triangulation (UIPT) in [17], see also [3] and [13]. Budd’s results in [13] apply to more general classes of random planar maps, known as Boltzmann planar maps, and as mentioned in the introduction of [13] they might allow the explicit calculation of other scaling constants arising when considering different metrics on these random graphs.

**The UIPT.** We can also state versions of our results on the UIPT, which is the random infinite lattice first discussed by Angel and Schramm [8] (in fact, Angel and Schramm did not consider type I triangulations, but that case is a special instance of the constructions in Stephenson [36]). The UIPT, which will be denoted by  $\mathcal{T}_\infty$ , is the local limit of uniformly distributed plane triangulations with  $n$  faces when  $n \rightarrow \infty$ . We can equip the vertex set of the UIPT with the usual graph distance  $d_{\text{gr}}$  or with a modified distance as above. For simplicity, let us consider only the first-passage percolation distance  $d_{\text{fpp}}$  defined as previously from i.i.d. edge weights (case 0.). For every  $r > 0$ , write  $B_r(\mathcal{T}_\infty)$  for the planar map obtained by keeping only those faces of  $\mathcal{T}_\infty$  that contain at least one vertex at graph distance strictly less than  $r$  from the root vertex, and define  $B_r^{\text{fpp}}(\mathcal{T}_\infty)$  analogously, replacing the graph distance by the first-passage percolation distance. Under the same assumptions on the weights, we prove (Theorem 2) that, for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow \infty} \mathbb{P} \left( \sup_{x,y \in V(B_r(\mathcal{T}_\infty))} |d_{\text{fpp}}(x,y) - \mathbf{c}_0 \cdot d_{\text{gr}}(x,y)| > \varepsilon r \right) = 0, \quad (4)$$

with the same constant  $\mathbf{c}_0$  as above. It follows that the inclusions

$$B_{(1-\varepsilon)r/\mathbf{c}_0}(\mathcal{T}_\infty) \subset B_r^{\text{fpp}}(\mathcal{T}_\infty) \subset B_{(1+\varepsilon)r/\mathbf{c}_0}(\mathcal{T}_\infty) \quad (5)$$

hold with probability tending to 1 as  $r \rightarrow \infty$ . In other words, large balls for the first-passage percolation distance are close to balls for the graph distance. Similar results hold for the graph distance or the Eden distance on the dual map of the UIPT (see Theorem 4). The particular case of the Eden model answers a question [35, Question 9.14] raised by Miller and Sheffield, who used the Eden growth model as a motivation for introducing the so-called Quantum Loewner Evolution of parameter  $(\frac{8}{3}, 0)$ . Notice that the value of  $\mathbf{c}_2$  had already been conjectured in [3, Remark 5].

**Comparison with other models of FPP.** Let us briefly discuss the connections between our results and the vast literature on (bond) first-passage percolation on regular lattices such as  $\mathbb{Z}^d$ . The fact (5) that our first-passage percolation distance grows balls that are close to “deterministic” balls for the graph distance can be seen as a counterpart to the classical shape theorem on regular lattices (see the survey papers [22, 10]). On the other hand, very little is known about the asymptotic shape of balls for first-passage percolation on  $\mathbb{Z}^d$ , and this shape is not expected to be the round ball since the anisotropy of  $\mathbb{Z}^d$  might persist in the limit. When the underlying lattice is the Delaunay triangulation of a standard Poisson point process on  $\mathbb{R}^2$ , rotational invariance is restored and the limit shape is a round ball as proved in [38]. However, the dilation factor of the ball, known as the time constant, remains out of reach.

Similarly to the case of Delaunay triangulations, our random setting of the UIPT is in a sense more “isotropic” than regular lattices (even though the graph is not embedded and so the meaning of rotational invariance is unclear) which explains intuitively why balls in the modified metric grow roughly like balls for the graph distance. It is remarkable that one can compute the values of the “time constants” in the particular cases 1. and 2. Let us also mention that first-passage percolation on random graphs has been considered in other models, either in the “dense” case (e.g. supercritical Erdős-Rényi random graphs) [39, Chapter 8] or in tree-like graphs [37]. In these cases too, the resulting first-passage percolation metric is typically proportional to the graph distance, with an explicit multiplicative constant.

**Extensions.** Our techniques should extend to much more general settings. In case 0. in particular, we expect that the assumption on the distribution of weights can be relaxed significantly. Assuming that this distribution is supported on a compact subinterval of  $(0, \infty)$  avoids a number of additional technicalities. Similarly, we can handle more general first-passage percolation distances on the dual map. We restricted our attention to cases 1. and 2. because these cases allow explicit calculations of the values of  $\mathbf{c}_1$  and  $\mathbf{c}_2$  (which seem hopeless in more general situations). Our techniques are robust and could probably handle more involved modifications of the graph distance as long as these modifications remain “local”. One such example suggested to us by Grégory Miermont is to consider the Riemannian metric associated with the Riemann surface structure of the map, which is obtained by gluing equilateral triangles according to the combinatorics of the map [15, 21].

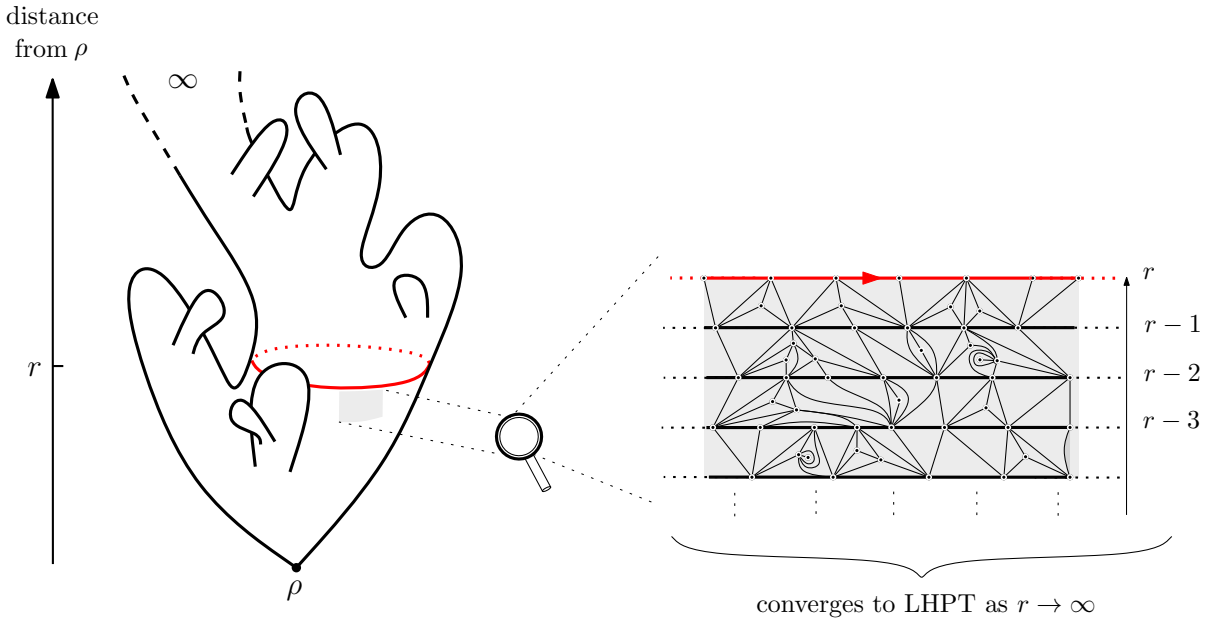
We also mention that similar techniques should be applicable to random *quadrangulations*, and we hope to address this setting in future work. The case of quadrangulations is especially interesting because of Tutte’s bijection between quadrangulations with  $n$  faces and general planar maps with  $n$  edges. We conjecture that our techniques can be used in combination with the results of [12] to verify that Tutte’s bijection is asymptotically an isometry.

## Ideas of proofs and methods

We conclude this introduction with a brief discussion of our methods. An important part of the paper (Sections 2 to 4) is devoted to geometric properties of the UIPT that are needed for the proofs of our

main theorems. Section 2 in particular discusses a decomposition of the UIPT into layers, which is closely related to the work of Krikun [24]. To explain this decomposition, we define, for every integer  $r \geq 1$ , the hull  $B_r^\bullet(\mathcal{T}_\infty)$  by adding to the ball  $B_r(\mathcal{T}_\infty)$  the finite connected components of its complement. We then call layer each set of the form  $B_r^\bullet(\mathcal{T}_\infty) \setminus B_{r-1}^\bullet(\mathcal{T}_\infty)$ . In each such layer we can distinguish special triangles, called downward triangles, which are in one-to-one correspondance with edges of the outer boundary  $\partial B_r^\bullet(\mathcal{T}_\infty)$  of the layer, see Fig. 3. It turns out that the configuration of downward triangles has a very nice branching structure which can be described in an explicit manner in terms of a critical offspring distribution  $\theta$  in the domain of attraction of a  $\frac{3}{2}$ -stable distribution. The configuration of downward triangles does not determine the UIPT, but it is easy to reconstruct the UIPT given this configuration: the holes remaining when one removes the downward triangles are filled in by independent Boltzmann triangulations with a boundary (called free triangulations in [8]).

The layer decomposition of the UIPT suggests to introduce two half-plane models, which we call the LHPT for lower half-plane triangulation and the UHPT for upper half-plane triangulation, and which are discussed in Section 3. Roughly speaking the LHPT corresponds to what one sees “below” the boundary of the hull  $B_r^\bullet(\mathcal{T}_\infty)$  when  $r$  is large (see Fig. 1 below). The UHPT, which was already discussed in [5, 7], is a kind of dual model to the LHPT, and arises as the local limit of (infinite) triangulations with a boundary when the size of the boundary tends to infinity. The model of interest for our purposes is the LHPT: The constants  $\mathbf{c}_0$ ,  $\mathbf{c}_1$  and  $\mathbf{c}_2$  arise from an application of Kingman’s ergodic theorem to the (modified) distance between the root vertex or the root face and the horizontal line at vertical coordinate  $-n$  in the LHPT (Propositions 18 and 25). However, certain estimates, concerning graph distances along the boundary in these half-plane models, are easier to derive in the UHPT model and can then be “transferred” to the LHPT using the relations between the two models. These estimates, which play a key role in the subsequent proofs, are discussed in Section 4.



**Figure 1:** Illustration of the link between the UIPT and the LHPT. The latter appears as the local limit of the hull of radius  $r$  of the UIPT seen from the boundary of the hull.

Section 5 is the most technical part of the paper. We concentrate on the first-passage percolation

distance  $d_{\text{fpp}}$  of case 0. and explain how to apply the asymptotics known in the LHPT model (resulting from the application of Kingman's subadditive ergodic theorem) in order to get information on the UIPT. The key Proposition 19 essentially shows that the  $d_{\text{fpp}}$ -distance between *any* vertex of the boundary  $\partial B_r^\bullet(\mathcal{T}_\infty)$  of the hull of radius  $r$  and the boundary  $\partial B_{r-\lfloor \eta r \rfloor}^\bullet(\mathcal{T}_\infty)$  of a smaller hull of radius  $r - \lfloor \eta r \rfloor$  is close to  $\mathbf{c}_0 \eta r$  when  $r$  is large, provided  $\eta > 0$  has been fixed small enough. The proof involves a delicate comparison argument with the LHPT model and makes use of the estimates for distances along the boundary derived in Section 4. From Proposition 19, it is not too hard to verify that, with high probability, any vertex of  $\partial B_r^\bullet(\mathcal{T}_\infty)$  is at  $d_{\text{fpp}}$ -distance approximately  $\mathbf{c}_0 r$  from the root vertex (Proposition 20). Our main results are then proved in Section 6 in the case 0. We use absolute continuity relations between the UIPT and finite triangulations to prove that the  $d_{\text{fpp}}$ -distance between the root vertex of  $\mathcal{T}_n$  and another vertex chosen uniformly at random in  $V(\mathcal{T}_n)$  is close to  $\mathbf{c}_0$  times the graph distance between the same two vertices, up to an error small in comparison with  $n^{1/4}$  (Proposition 21). The convergence (1) follows easily. We can then return to the UIPT and deduce (4) from (1) and another application of the absolute continuity relations between the UIPT and finite triangulations. Finally Section 7 explains the modifications needed to extend our results to cases 1. and 2. concerning the dual graph distance  $d_{\text{gr}}^\dagger$  and the Eden distance  $d_{\text{Eden}}^\dagger$ . As mentioned above, once the analog of (5) has been proved in these cases, the values of  $\mathbf{c}_1$  and  $\mathbf{c}_2$  can be deduced from our previous work [17] on the peeling process on random triangulations.

**Acknowledgments:** We thank Timothy Budd and Grégory Miermont for stimulating discussions.

## 2 Skeleton decomposition of random triangulations

In this section we present the skeleton decomposition of (type I) triangulations. This is closely related to the work of Krikun [24, 23]. Using explicit enumeration formulas found in [25] we are able to give simple expressions for the law of the skeleton of finite random triangulations.

We focus on the case of type I triangulations, where both loops and multiple edges are allowed. Many of our arguments can be adapted to the case of type II triangulations, but we do not discuss this here. As usual, all the planar maps that are considered in this work are rooted, i.e. given with a distinguished oriented edge called the root edge, whose initial vertex is called the root vertex. A triangulation  $t$  with a boundary is a rooted planar map such that all faces are triangles, except for the face incident to the right of the root edge, which must be a simple face (its boundary is a simple cycle). The latter face will be called the bottom face of the triangulation, and its boundary  $\partial t$  is called the bottom cycle. If the length of the bottom cycle is  $p$ , we speak of a triangulation of the  $p$ -gon. The height of a vertex  $x \in V(t)$  is the minimal graph distance between  $x$  and a vertex of the bottom cycle.

### 2.1 Enumeration background

For every  $p \geq 1$  and  $n \geq 0$ , we let  $\mathbb{T}_{n,p}$  be the set of all (type I) triangulations of the  $p$ -gon with  $n$  inner vertices. We list here the enumeration results that we will need. These results can be found in Krikun [25] (Krikun uses the number of edges as the size parameter and in order to apply his formulas we note that a triangulation of the  $p$ -gon with  $n$  inner vertices has  $3n + 2p - 3$  edges). The set  $\mathbb{T}_{0,1}$  is empty, and, for  $n \geq 1$  and  $p \geq 1$ , or  $n = 0$  and  $p \geq 2$ , we have

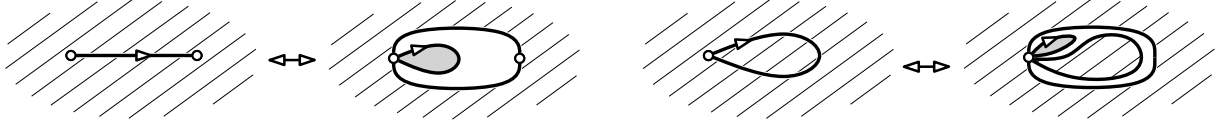
$$\#\mathbb{T}_{n,p} = 4^{n-1} \frac{p(2p)!(2p+3n-5)!!}{(p!)^2 n! (2p+n-1)!!} \underset{n \rightarrow \infty}{\sim} C(p) (12\sqrt{3})^n n^{-5/2}, \quad (6)$$

with

$$C(p) = \frac{3^{p-2} p (2p)!}{4\sqrt{2\pi} (p!)^2} \underset{p \rightarrow \infty}{\sim} \frac{1}{36\pi\sqrt{2}} \sqrt{p} 12^p, \quad (7)$$

where we write  $a(p) \sim b(p)$  if the ratio  $a(p)/b(p)$  tends to 1. Note that formula (6) gives  $\#\mathbb{T}_{0,2} = 1$ , with the usual convention  $(-1)!! = 1$ . This formula holds because we consider the rooted planar map consisting of a single (oriented) edge as a trivial triangulation of the 2-gon. This “edge-triangulation” plays an important role as it will be used later to “glue” the two sides of a 2-gon.

Consider a (rooted) plane triangulation  $t$  with  $n$  vertices ( $n \geq 3$ ). If we split the root edge of  $t$  into a double edge, then add a loop inside the region bounded by this double edge, and root the resulting map on this new loop in clockwise direction, we obtain a triangulation of the 1-gon with  $n - 1$  internal vertices. The way the new loop is added should be clear from Fig. 2, and we note that the construction works as well in the case where the initial root edge is itself a loop, as shown in the right part of Fig. 2. The previous construction yields, for every  $n \geq 3$ , a bijection between  $\mathbb{T}_{n-1,1}$  and the set of all rooted plane triangulations with  $n$  vertices. Hence, we may and will often see rooted plane triangulations as triangulations of the 1-gon.



**Figure 2:** Transforming the root edge into a loop (in the right part, the case where the initial root edge is a loop) provides a bijection between  $\mathbb{T}_{n-1,1}$  and the set of all rooted plane triangulations with  $n$  vertices.

Fix  $p \geq 1$  and set  $Z(p) = \sum_{n \geq 0} (12\sqrt{3})^{-n} \#\mathbb{T}_{n,p}$ . Note that the series converges because of the asymptotics in (6). The (critical) Boltzmann distribution on triangulations of the  $p$ -gon is the probability measure on  $\cup_{n \geq 0} \mathbb{T}_{n,p}$  that assigns mass  $(12\sqrt{3})^{-n}/Z(p)$  to every element of  $\mathbb{T}_{n,p}$ , for every  $n \geq 0$ . We have the exact formulas (see [7, Section 2.2])

$$Z(p) = \frac{6^p (2p-5)!!}{8\sqrt{3} p!} \quad \text{if } p \geq 2, \quad Z(1) = \frac{2 - \sqrt{3}}{4}. \quad (8)$$

The generating series of  $Z(p)$  can also be computed explicitly from [25, formula (4)] and an appropriate change of variables (we omit the details):

$$\sum_{p \geq 0} Z(p+1) z^p = \frac{1}{2} + \frac{(1 - 12z)^{3/2} - 1}{24\sqrt{3}z}. \quad (9)$$

From the explicit formula (8) and Stirling’s formula, we have

$$Z(p+1) \underset{p \rightarrow \infty}{\sim} \frac{\sqrt{3}}{8\sqrt{\pi}} 12^p p^{-5/2}, \quad (10)$$

**Lemma 1.** *There exists a constant  $c > 0$  such that, for every  $n \geq 1$  and  $p \geq 1$ , we have*

$$\#\mathbb{T}_{n,p} \leq c \cdot C(p) n^{-5/2} (12\sqrt{3})^n.$$

*Furthermore, for every choice of  $\alpha > 0$ , there exists a constant  $c' = c'(\alpha) > 0$  such that, for every  $n \geq 1$  and for all integers  $p$  with  $1 \leq p \leq \alpha\sqrt{n}$ , we have*

$$\#\mathbb{T}_{n,p} \geq c' \cdot C(p) n^{-5/2} (12\sqrt{3})^n.$$



*Proof.* We start with the first assertion. Note that this estimate does not immediately follow from (6) because we want  $c$  to be independent of  $p$ . However, (6) allows us to restrict our attention to the case  $p \geq 2$ . Using Stirling's formula and the fact that  $c_1 \sqrt{n}(n/e)^{n/2} \leq n!! \leq c_2 \sqrt{n}(n/e)^{n/2}$  for every  $n \geq 1$ , for some positive constants  $c_1$  and  $c_2$ , we get, with a constant  $c$  that may vary from line to line,

$$\begin{aligned}
4^{n-1} \frac{p(2p)!(2p+3n-5)!!}{(p!)^2 n! (2p+n-1)!!} &\leq c \cdot 4^n p \frac{4^p}{\sqrt{p}} \sqrt{\frac{2p+3n-5}{2p+n-1}} \frac{1}{\sqrt{n}} \left( \frac{(2p+3n-5)^{2p+3n-5}}{n^{2n}(2p+n-1)^{2p+n-1}} \right)^{1/2} \\
&\leq c \cdot \frac{4^n}{\sqrt{n}} \sqrt{p} 4^p \left( \frac{(3n)^{2p+3n-5} \left(\frac{2p}{3n} + 1 - \frac{5}{3n}\right)^{2p+3n-5}}{n^{2p+3n-1} \left(\frac{2p}{n} + 1 - \frac{1}{n}\right)^{2p+n-1}} \right)^{1/2} \\
&\leq c \cdot \frac{(12\sqrt{3})^n}{n^{5/2}} \sqrt{p} 12^p \left( \frac{\left(\frac{2p}{3n} + 1 - \frac{5}{3n}\right)^{2p+3n-5}}{\left(\frac{2p}{n} + 1 - \frac{1}{n}\right)^{2p+n-1}} \right)^{1/2} \\
&\leq c \cdot C(p) \frac{(12\sqrt{3})^n}{n^{5/2}} \left( \frac{\left(\frac{2p-1}{3n} + 1\right)^{\frac{2p-1}{n}+3}}{\left(\frac{2p-1}{n} + 1\right)^{\frac{2p-1}{n}+1}} \right)^{n/2}.
\end{aligned}$$

To complete the proof of the first assertion, it suffices to verify that the quantity inside the big parentheses in the last display is smaller than 1 for any  $n \geq 1$  and  $p \geq 2$ . To this end, take  $x = (p - \frac{1}{2})/n \geq 0$  and note that the function  $f(x) = (2x/3 + 1)^{2x+3}(2x+1)^{-2x-1}$  is bounded above by 1 on  $\mathbb{R}_+$  (setting  $u = 2x+1 \geq 1$ , and taking logarithms, we need to see that  $(u+2)(\log(u+2) - \log 3) - u \log u \leq 0$ , which we get by differentiating this function of  $u$ ).

To get the second assertion of the lemma, we use similar arguments to arrive at the lower bound

$$4^{n-1} \frac{p(2p)!(2p+3n-5)!!}{(p!)^2 n! (2p+n-1)!!} \geq c' \cdot C(p) \frac{(12\sqrt{3})^n}{n^{5/2}} \left( \frac{\left(\frac{2p}{3n} + 1 - \frac{5}{3n}\right)^{2p+3n-5}}{\left(\frac{2p}{n} + 1 - \frac{1}{n}\right)^{2p+n-1}} \right)^{1/2},$$

and elementary considerations show that, under our condition  $1 \leq p \leq \alpha\sqrt{n}$ , the ratio

$$\frac{\left(\frac{2p}{3n} + 1 - \frac{5}{3n}\right)^{2p+3n-5}}{\left(\frac{2p}{n} + 1 - \frac{1}{n}\right)^{2p+n-1}}$$

is bounded below by a positive constant depending on  $\alpha$ .  $\square$

## 2.2 Triangulations of the cylinder and their skeleton decomposition

In this section, we discuss a special class of planar maps, which we call triangulations of the cylinder. The reason for considering this class is the fact that hulls of triangulations with a boundary will be triangulations of the cylinder. We then describe these objects via a *skeleton decomposition* which was first discussed in Krikun [24, 23] in the case of type II triangulations and quadrangulations.

**Definition 1.** Let  $r \geq 1$  be an integer. A triangulation of the cylinder of height  $r$  is a rooted planar map such that all faces are triangles except for two distinguished faces called respectively the bottom face and the top face, and such that the following properties hold. The boundaries of the bottom face and of the top face are disjoint simple cycles. The boundary of the bottom face contains the root edge, which is oriented in such a way that the bottom face lies on its right. Finally, every vertex incident to the top face is at graph distance exactly  $r$  from the boundary of the bottom face, and every edge incident to the top face is also incident to a triangle whose third vertex is at distance  $r-1$  from the bottom face.

If  $\Delta$  is a triangulation of the cylinder of height  $r$ , the bottom cycle is again denoted by  $\partial\Delta$  and the top cycle (boundary of the top face) of  $\Delta$  is denoted by  $\partial^*\Delta$ .



Let  $\Delta$  be a fixed triangulation of the cylinder of height  $r$ . We let  $p \geq 1$  be the length of the bottom cycle  $\partial\Delta$ , and let  $q \geq 1$  be the length of the top cycle  $\partial^*\Delta$ . We will now describe a skeleton decomposition that encodes the triangulation  $\Delta$  via an ordered forest of  $q$  (rooted) plane trees with maximal height  $r$ , and a collection, indexed by the vertices of the forest at height strictly less than  $r$ , of triangulations with a boundary. To describe this decomposition, we first need to introduce some notation. For  $1 \leq j < r$ , the ball  $B_j(\Delta)$  is defined as the union of all faces of  $\Delta$  that are incident to a vertex at graph distance strictly less than  $j$  from the bottom cycle, and the hull  $B_j^\bullet(\Delta)$  is obtained by adding to the ball  $B_j(\Delta)$  all connected components of its complement except for the one containing the top cycle. It is an easy exercise to see that  $B_j^\bullet(\Delta)$  is then a triangulation of the cylinder of height  $j$ , and we let  $\partial_j\Delta = \partial^*B_j^\bullet(\Delta)$  denote its top cycle. By convention,  $\partial_0\Delta = \partial\Delta$  is the bottom cycle of  $\Delta$  and  $\partial_r\Delta = \partial^*\Delta$  is the top cycle of  $\Delta$ . We may and will assume that  $\Delta$  is drawn in the plane in such a way that the top face is the unbounded face, as in Fig. 3, and we then orient all cycles  $\partial_j\Delta$  in clockwise order (for  $j = 0$  this is consistent with the orientation of the root edge).

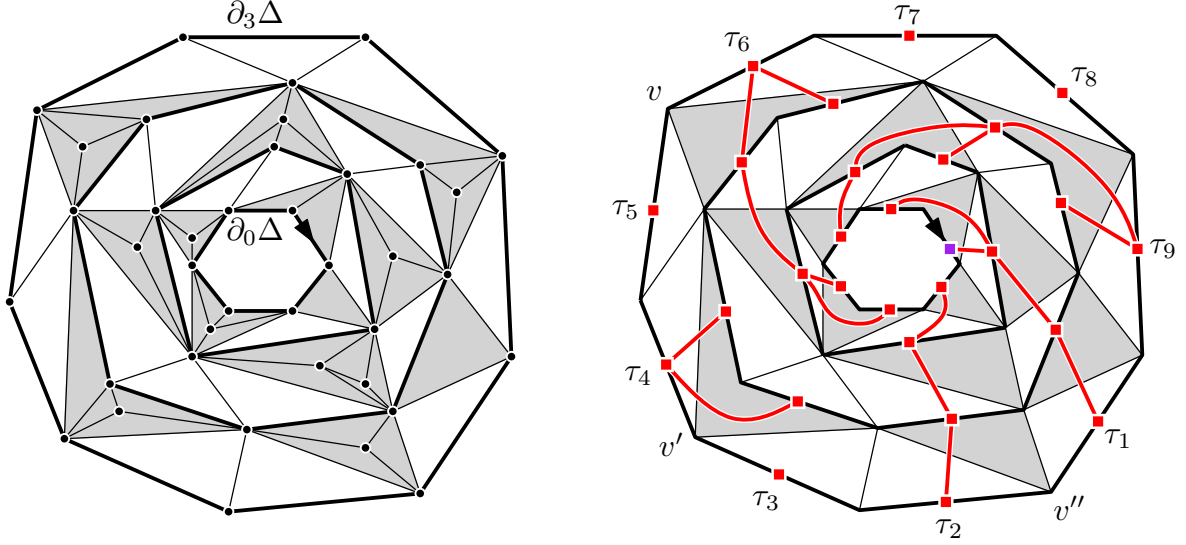
If  $k \in \{1, 2, \dots, r\}$ , every edge of  $\partial_k\Delta$  is incident to (exactly) one triangle whose third vertex belongs to  $\partial_{k-1}\Delta$ . We call such triangles the downward triangles at height  $k$ . These triangles are in one-to-one correspondence with the edges of  $\partial_k\Delta$ . Let  $E_d(\Delta)$  be the collection of all edges of  $\Delta$  that belong to one of the cycles  $\partial_j\Delta$  for  $0 \leq j \leq r$ . We define a genealogical order on the set  $E_d(\Delta)$  by saying that, for every  $k \in \{1, \dots, r\}$ , an edge  $e$  of  $\partial_k\Delta$  is the “parent” of an edge  $e'$  of  $\partial_{k-1}\Delta$  if the downward triangle associated with  $e$  is the first one that one encounters when turning around  $\partial_{k-1}\Delta$  inside  $\Delta \setminus B_{k-1}(\Delta)$  in clockwise order, starting from the middle of the edge  $e'$ . The definition of this genealogical order should be clear from the right part of Fig. 3.

Thanks to the planar structure of  $\Delta$ , these genealogical relations lead to a forest of  $q$  plane trees, whose vertices are (in one-to-one correspondence with) the edges belonging to  $E_d(\Delta)$ , and which are rooted at the edges of  $\partial_r\Delta$ . The maximal height in the forest is  $r$ , and a vertex at height (distance from the ancestor)  $r - j$ , for  $0 \leq j \leq r$ , corresponds to an edge of  $\partial_j\Delta$ . See Fig. 3. We write  $\tau_1, \tau_2, \dots, \tau_q$  for the trees in the forest listed around  $\partial^*\Delta$  in clockwise order, in such a way that  $\tau_1$  is the tree containing the vertex corresponding to the root edge of  $\Delta$ . Note that  $\tau_1$  is a tree with height  $r$  and a distinguished vertex (the root edge of  $\Delta$ ) at height  $r$ .

The forest  $(\tau_1, \tau_2, \dots, \tau_q)$  does not give enough information to reconstruct the triangulation  $\Delta$ . Indeed this forest only characterizes the configuration of downward triangles and, informally, we need to “fill in” the holes left by these triangles. More precisely, if we remove all the downward triangles from  $\Delta$ , we are left with the top and bottom faces and a collection of “slots”. If  $e$  is an edge of  $\partial_k\Delta$ , where  $1 \leq k \leq r$ , the associated slot is bounded by the edges of  $\partial_{k-1}\Delta$  that are children of  $e$  and by two “vertical” edges connecting the initial vertex of  $e$  (recall that  $\partial_k\Delta$  is oriented in clockwise order) to vertices of  $\partial_{k-1}\Delta$  (if  $e$  has no child, these two vertical edges may, or may not, be glued in a single edge, see Fig. 4). We may assign a root edge to the boundary of each slot, by deciding that the root edge of the slot associated with an edge  $e$  of  $\partial_k\Delta$  is the vertical edge, oriented so that its initial vertex is on  $\partial_{k-1}\Delta$ , which is incident on its right to the downward triangle associated with  $e$ . Then the slot associated with  $e$  is filled in by a well-defined triangulation of the  $(c_e + 2)$ -gon, where  $c_e \geq 0$  is the number of children of  $e$ . Note that when  $c_e = 0$  it may happen that the slot is filled in by the edge-triangulation, which just means that the two vertical edges are glued together.

Say that a forest  $\mathcal{F}$  with a distinguished vertex is  $(p, q, r)$ -admissible if

- (i) the forest consists of an ordered sequence  $(\tau_1, \tau_2, \dots, \tau_q)$  of  $q$  (rooted) plane trees,
- (ii) the maximal height of these trees is  $r$ ,



**Figure 3:** The skeleton decomposition of a triangulation of the cylinder of height 3. We have chosen to show a triangulation without multiple edges or loops for the sake of clarity of the figure. The downward triangles are represented in white and the other triangles are in grey in the left part of the figure. In the right part, we have erased the edges that are not incident to downward triangles (except for those of  $\partial_0 \Delta = \partial \Delta$ ) so that the slots appear in grey. The forest of trees associated with the triangulation is in red in the right part of the figure (notice that the trees  $\tau_3, \tau_5, \tau_7, \tau_8$  are trivial trees consisting only of the root).

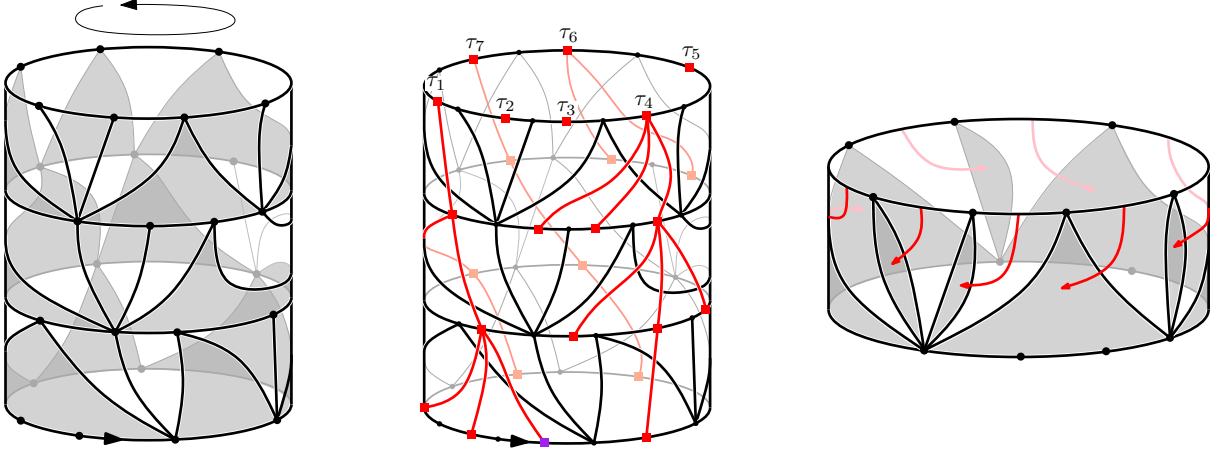
- (iii) the total number of vertices of the forest at generation  $r$  is  $p$ ,
- (iv) the distinguished vertex has height  $r$ ,
- (v) the distinguished vertex is in  $\tau_1$ .

If  $\mathcal{F}$  is a  $(p, q, r)$ -admissible forest, we write  $\mathcal{F}^*$  for the set all vertices of  $\mathcal{F}$  at height strictly less than  $r$ .

The preceding decomposition yields a bijection between, on the one hand, triangulations  $\Delta$  of the cylinder of height  $r$  with a bottom cycle of length  $p$  and a top cycle of length  $q$ , and on the other hand, pairs consisting of a  $(p, q, r)$ -admissible pointed forest  $\mathcal{F}$  and a collection  $(M_v)_{v \in \mathcal{F}^*}$  such that, for every  $v \in \mathcal{F}^*$ ,  $M_v$  is a triangulation of the  $(c_v + 2)$ -gon, if  $c_v$  stands for the number of children of  $v$  in  $\mathcal{F}$ . We call this bijection the skeleton decomposition and we say that  $\mathcal{F}$  is the skeleton of the triangulation  $\Delta$ .

**Remark 1.** *Our skeleton decomposition is slightly simpler than the original one in Krikun [24]: Because we deal with type I triangulations, where loops are allowed, we do not need to exclude the case considered in [24, Lemma 2.2].*

**Left-most geodesics.** Let  $x$  be a vertex of  $\partial_j \Delta$ , where  $1 \leq j \leq r$ . We define the left-most geodesic from  $x$  to the bottom cycle in the following way. We first observe that half-edges incident to  $x$  can be enumerated in clockwise order around  $x$ , starting from the half-edge of  $\partial_j \Delta$  whose initial vertex is  $x$  (recall that we have also oriented the cycles  $\partial_k \Delta$ ). The first edge on the left-most geodesic from  $x$  is the last edge connecting  $x$  to  $\partial_{j-1} \Delta$  arising in this enumeration. The path is then continued by the obvious induction.



**Figure 4:** Left, a representation of a triangulation without multiple edges of the cylinder of height 3 with the slots in grey. In the middle, the construction of the associated forest. Right, the slots of a triangulation of the cylinder of height 1 (possibly having multiple edges): when an edge of the top boundary has no child, the corresponding slot is bounded by a double edge, which may be glued into a single one if the slot is filled in by the edge-triangulation.

Left-most geodesics from distinct vertices may coalesce, and more precisely we have the following property. Let  $u$  and  $v$  be two distinct vertices of  $\partial^* \Delta$ . If  $\mathcal{F}$  is the skeleton of  $\Delta$ , let  $\mathcal{F}'$  be the subforest of  $\mathcal{F}$  consisting of the trees rooted at the edges of  $\partial^* \Delta$  that belong to the path going from  $u$  to  $v$  in clockwise order in  $\partial^* \Delta$ , and let  $\mathcal{F}''$  consist of the trees of  $\mathcal{F}$  that are not in  $\mathcal{F}'$ . Then, for every  $k \in \{1, \dots, r\}$ , the left-most geodesics starting respectively from  $u$  and from  $v$  merge before step  $k$  (possibly at step  $k$ ) if and only if at least one of the two forests  $\mathcal{F}'$  and  $\mathcal{F}''$  has height strictly smaller than  $k$ . We leave the easy verification to the reader (as an illustration, the reader may verify in Fig. 3 that the left-most geodesics starting respectively from  $v$  and  $v'$  coalesce at step 2, but the left-most geodesics starting respectively from  $v'$  and  $v''$  do not coalesce).

### 2.3 Hulls as triangulations of the cylinder

Let  $t$  be a triangulation of the  $p$ -gon, where  $p \geq 1$ . For every integer  $r \geq 1$ , the ball  $B_r(t)$  is the planar map obtained as the union of all faces of  $t$  that are incident to a vertex at distance less than or equal to  $r - 1$  from the bottom cycle. Clearly, the graph distance between any vertex of  $B_r(t)$  and the bottom cycle is at most  $r$ , and this graph distance is exactly  $r$  if the vertex is in the boundary of a connected component of the complement of  $B_r(t)$  in the sphere.

We now let  $o$  be a distinguished vertex of  $t$ . Then  $\bar{t} := (t, o)$  is a pointed (and rooted) triangulation of the  $p$ -gon. Let  $h$  be the height of  $o$  (recall that this is the minimal distance between  $o$  and a vertex of the bottom cycle). Suppose that  $r < h$ , so that the distinguished vertex  $o$  lies in the complement of  $B_r(t)$ . The hull  $B_r^\bullet(\bar{t})$  is obtained by adding to  $B_r(t)$  all connected components of the complement of  $B_r(t)$ , except for the one containing  $o$ . Then under the preceding assumptions, it is easy to see that the hull  $B_r^\bullet(\bar{t})$  is a triangulation of the cylinder of height  $r$ , in the sense of Definition 1. For future reference, we note that  $B_j^\bullet(B_r^\bullet(\bar{t})) = B_j^\bullet(\bar{t})$ , for  $1 \leq j < r$ .

## 2.4 The skeleton decomposition of random triangulations

We now consider random triangulations. We fix  $p \geq 1$  and, for every  $n \geq 1$ , we let  $\mathcal{T}_n^{(p)}$  be uniformly distributed over the set  $\mathbb{T}_{n,p}$  of all triangulations of the  $p$ -gon with  $n$  inner vertices. We also write  $\overline{\mathcal{T}}_n^{(p)}$  for the pointed triangulation obtained by choosing an inner vertex uniformly at random in  $\mathcal{T}_n^{(p)}$ . For every integer  $r \geq 1$ , we can make sense of the hull  $B_r^\bullet(\overline{\mathcal{T}}_n^{(p)})$  provided that the distance between the distinguished vertex and the bottom cycle is at least  $r + 1$ . For definiteness, if the latter condition does not hold, we define the hull  $B_r^\bullet(\overline{\mathcal{T}}_n^{(p)})$  as the edge-triangulation. Our goal is to evaluate the asymptotics as  $n \rightarrow \infty$  of the probability that the hull  $B_r^\bullet(\overline{\mathcal{T}}_n^{(p)})$  is equal to a given triangulation  $\Delta$  of the cylinder of height  $r$ , in terms of the skeleton decomposition of  $\Delta$ .

Let  $r \geq 1$ , and let  $\Delta$  be a triangulation of the cylinder of height  $r$ . Write  $p$  for the length of the bottom cycle of  $\Delta$  and  $q$  for the length of the top cycle. Also let  $N$  be the total number of vertices of  $\Delta$ . By the skeleton decomposition of Section 2.2, we can associate with  $\Delta$  a  $(p, q, r)$ -admissible forest  $\mathcal{F} = (\tau_1, \dots, \tau_q)$ . As in Section 2.2, we write  $(M_v)_{v \in \mathcal{F}^*}$  for the triangulations with a boundary filling in the slots in  $\Delta$ . We also let  $\text{Inn}(M_v)$  stand for the number of inner vertices of  $M_v$ , for every  $v \in \mathcal{F}^*$ . Recall the constants  $C(p)$  and  $Z(p)$  defined in (7) and (8).

**Lemma 2.** *We have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(p)}) = \Delta) = \frac{12^{-q} C(q)}{12^{-p} C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v) \times \prod_{v \in \mathcal{F}^*} \frac{(12\sqrt{3})^{-\text{Inn}(M_v)}}{Z(c_v + 2)}$$

where  $c_v$  is the number of children of  $v$  in the forest  $\mathcal{F}$ , and  $\theta$  is the critical offspring distribution whose generating function  $g_\theta$  is given by

$$g_\theta(x) = \sum_{i=0}^{\infty} x^i \theta(i) = 1 - \left(1 + \frac{1}{\sqrt{1-x}}\right)^{-2}, \quad x \in [0, 1].$$

In particular we have

$$\theta(k) \underset{k \rightarrow \infty}{\sim} \frac{3}{2\sqrt{\pi}} k^{-5/2}. \quad (11)$$

*Proof.* In this proof, we set  $\rho = 12\sqrt{3}$  and  $\alpha = 12$  to simplify notation. We observe that the property  $B_r^\bullet(\overline{\mathcal{T}}_n^{(p)}) = \Delta$  holds if and only if  $\mathcal{T}_n^{(p)}$  is obtained from  $\Delta$  by gluing an arbitrary triangulation of the  $q$ -gon with  $n - (N - p)$  inner vertices on the top cycle of  $\Delta$  (to perform this gluing, we may assume that the top cycle is rooted at the root of the first forest in  $\mathcal{F}$ ) and if the distinguished vertex of  $\overline{\mathcal{T}}_n^{(p)}$  is chosen among the inner vertices of the glued triangulation. It follows that, for  $n \geq N - p$ ,

$$\mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(p)}) = \Delta) = \frac{\#\mathbb{T}_{n-(N-p),q}}{\#\mathbb{T}_{n,p}} \times \frac{n - (N - p)}{n}. \quad (12)$$

Using (6), we now get

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(p)}) = \Delta) = \frac{C(q)}{C(p)} \rho^{-N+p}. \quad (13)$$

Notice that the total number of vertices of  $\Delta$  can be decomposed as  $N = S + \sum_{v \in \mathcal{F}^*} \text{Inn}(M_v)$  where  $S = \#\mathbb{E}_d(\Delta)$  is the total number of vertices in the cycles  $\partial_j \Delta$ ,  $0 \leq j \leq r$ , with the notation of Section 2.2. We have

$$S = \sum_{i=1}^q \#\tau_i = q + \sum_{v \in \mathcal{F}^*} c_v.$$

Hence the right-hand side of (13) is also equal to

$$\frac{C(q)}{C(p)} \rho^{p-q} \prod_{v \in \mathcal{F}^*} \rho^{-\ln(M_v)} \rho^{-c_v}.$$

We then observe that  $\sum_{v \in \mathcal{F}^*} (c_v - 1) = p - q$  and so we can multiply the quantity in the last display by

$$\left(\frac{\alpha}{\rho}\right)^{p-q-\sum_{v \in \mathcal{F}^*} (c_v - 1)}$$

to get that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(B_r^\bullet(\bar{T}_n^{(p)}) = \Delta) &= \frac{\alpha^{-q} C(q)}{\alpha^{-p} C(p)} \prod_{v \in \mathcal{F}^*} \left( \rho^{-1} \alpha^{-c_v+1} \rho^{-\ln(M_v)} \right) \\ &= \frac{\alpha^{-q} C(q)}{\alpha^{-p} C(p)} \prod_{v \in \mathcal{F}^*} \left( \theta(c_v) \cdot \frac{\rho^{-\ln(M_v)}}{Z(c_v + 2)} \right), \end{aligned} \quad (14)$$

where we have set

$$\theta(k) = \frac{1}{\rho} \alpha^{-k+1} Z(k+2), \quad \text{for every } k \geq 0. \quad (15)$$

Recall that  $\rho = 12\sqrt{3}$  and  $\alpha = 12$  so that, using (8) and (9), we get for  $0 < x \leq 1$ ,

$$\begin{aligned} \sum_{k=0}^{\infty} x^k \theta(k) &= \frac{\alpha}{\rho} \sum_{k=0}^{\infty} \left(\frac{x}{\alpha}\right)^k Z(k+2) \\ &= \frac{\alpha}{\rho} \sum_{i=1}^{\infty} \left(\frac{x}{\alpha}\right)^{i-1} Z(i+1) \\ &= \frac{\alpha^2}{x\rho} \left( \sum_{i=0}^{\infty} \left(\frac{x}{\alpha}\right)^i Z(i+1) - Z(1) \right) \\ &= \frac{4\sqrt{3}}{x} \left( \frac{1}{2} + \frac{(1-x)^{3/2} - 1}{2\sqrt{3}x} - \frac{2 - \sqrt{3}}{4} \right) \\ &= \frac{2(1-x)^{3/2} + 3x - 2}{x^2} \\ &= 1 - \left( 1 + \frac{1}{\sqrt{1-x}} \right)^{-2}. \end{aligned}$$

From this explicit formula for the generating function, one gets that  $\theta$  is a probability distribution with mean 1. Finally the asymptotics (11) are derived from the formula (15) for  $\theta(k)$  and (10). This completes the proof.  $\square$

**Remark 2.** *It is interesting to notice that although we consider a slightly different model, namely type I triangulations, we recover the same offspring distribution  $\theta$  in the case of type II triangulations [24]. This may be explained geometrically by the relations between type I and type II triangulations.*

The distribution  $\theta$  plays an important role in this work. It will be convenient to write  $Y = (Y_r)_{r \geq 0}$  for a Galton–Watson process with offspring distribution  $\theta$  that starts from  $k$  under the probability measure  $\mathbb{P}_k$ . Then, for every integer  $r \geq 1$ , the generating function of the distribution of  $Z_r$  under  $\mathbb{P}_1$  is the  $r$ -th iterate  $g_\theta^{(r)}$  of  $g_\theta$ . A simple induction shows that, for every integer  $r \geq 1$ , for every  $x \in [0, 1)$ ,

$$\mathbb{E}_1[x^{Y_r}] = g_\theta^{(r)}(x) = 1 - \left( r + \frac{1}{\sqrt{1-x}} \right)^{-2}. \quad (16)$$

The function  $g_\theta^{(r)}$  can be extended to a holomorphic function on  $\mathbb{C} \setminus [1, \infty)$ . For this extension, we have

$$g_\theta^{(r)}(z) - z = (1 - z) \left( 1 - \frac{1}{1 + r\sqrt{1 - z}} \right) \left( 1 + \frac{1}{1 + r\sqrt{1 - z}} \right) \sim 2r(1 - z)^{3/2},$$

as  $z \rightarrow 1$ ,  $z \in \mathbb{C} \setminus [1, \infty)$ . By a well-known result of singularity analysis (see Corollary VI.1 p.392 in [20]), it follows that

$$\mathbb{P}_1(Y_r = k) \underset{k \rightarrow \infty}{\sim} \frac{3r}{2\sqrt{\pi}} k^{-5/2}. \quad (17)$$

Of course we recover (11) when  $r = 1$ .

We let  $\mathbb{F}_{p,q,r}$  be the set of all  $(p, q, r)$ -admissible forests. We will also consider the set  $\mathbb{F}'_{p,q,r}$  of all (pointed) forests satisfying the same properties as  $(p, q, r)$ -admissible forests, except that we do not require property (v) saying that the distinguished vertex belongs to the first tree in the forest. Finally, we will write  $\mathbb{F}''_{p,q,r}$  for the set of all (non-pointed) forests satisfying the same properties (i)–(iii) as  $(p, q, r)$ -admissible forests except that no special vertex is distinguished (and so properties (iv) and (v) become irrelevant).

**Lemma 3.** *For every  $p \geq 1$  and  $r \geq 1$ ,*

$$\sum_{q=1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}_{p,q,r}} \frac{12^{-q}C(q)}{12^{-p}C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v) = 1.$$

*Proof.* We start with some simple observations. If  $\mathcal{F}' \in \mathbb{F}'_{p,q,r}$ , we can obtain a forest  $\mathbf{p}(\mathcal{F}') \in \mathbb{F}_{p,q,r}$  by cyclically permuting the trees so that the first one will contain the distinguished vertex, and if  $\mathcal{F} \in \mathbb{F}_{p,q,r}$  there are exactly  $q$  choices of  $\mathcal{F}' \in \mathbb{F}'_{p,q,r}$  such that  $\mathbf{p}(\mathcal{F}') = \mathcal{F}$ . Hence, we have also

$$\sum_{q=1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}_{p,q,r}} \frac{12^{-q}C(q)}{12^{-p}C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v) = \sum_{q=1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}'_{p,q,r}} \frac{1}{q} \frac{12^{-q}C(q)}{12^{-p}C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v).$$

Then, with every  $\mathcal{F}' \in \mathbb{F}'_{p,q,r}$ , we can associate  $\mathcal{F}'' \in \mathbb{F}''_{p,q,r}$  by simply “forgetting” the distinguished vertex, and each  $\mathcal{F}'' \in \mathbb{F}''_{p,q,r}$  corresponds through this operation to  $p$  possible choices of  $\mathcal{F}' \in \mathbb{F}'_{p,q,r}$ . Hence,

$$\sum_{q=1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}'_{p,q,r}} \frac{1}{q} \frac{12^{-q}C(q)}{12^{-p}C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v) = \sum_{q=1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}''_{p,q,r}} \frac{p}{q} \frac{12^{-q}C(q)}{12^{-p}C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v) = \sum_{q=1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}''_{p,q,r}} \frac{h(q)}{h(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v), \quad (18)$$

where using (7) we put for every  $k \geq 1$ ,

$$h(k) = 4^{-k} \binom{2k}{k} = 36\sqrt{2\pi} \frac{12^{-k}C(k)}{k}.$$

The right-hand side of (18) can also be written as

$$\sum_{q=1}^{\infty} \frac{h(q)}{h(p)} \mathbb{P}_q(Y_r = p)$$

with the notation introduced before the statement of the lemma. So in order to prove the lemma, we need to verify that, for every  $p \geq 1$ ,

$$\sum_{q=1}^{\infty} h(q) \mathbb{P}_q(Y_r = p) = h(p). \quad (19)$$

This is equivalent to saying that  $(h(k))_{k \geq 1}$  is an infinite stationary measure for the Galton–Watson process  $Y$ . To see this, set, for every  $x \in [0, 1)$ ,

$$\Pi(x) = \sum_{k=1}^{\infty} h(k) x^k = \frac{1}{\sqrt{1-x}} - 1.$$

To verify that  $(h(k))_{k \geq 1}$  is a stationary distribution for  $Y$ , it is enough [9, Chapter II] to check that  $\Pi(g_\theta(x)) - \Pi(g_\theta(0)) = \Pi(x)$ , for every  $x \in [0, 1)$ . This follows from the explicit formulas for  $g_\theta$  and  $\Pi$ .  $\square$

For fixed integers  $p \geq 1$  and  $r \geq 1$ , define

$$\mathbb{F}_{p,r} := \bigcup_{q=1}^{\infty} \mathbb{F}_{p,q,r}.$$

Also write  $\mathbb{C}_{p,r}$  for the (countable) set of all triangulations of the cylinder of height  $r$  with a bottom cycle of length  $p$ .

Lemma 3 allows us to define a probability measure  $\mathbf{P}_{p,r}$  on  $\mathbb{F}_{p,r}$  by setting, for every forest  $\mathcal{F} \in \mathbb{F}_{p,q,r}$ ,

$$\mathbf{P}_{p,r}(\mathcal{F}) := \frac{12^{-q} C(q)}{12^{-p} C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v). \quad (20)$$

We then define a probability measure  $\mathbb{P}_{p,r}$  on the set  $\mathbb{C}_{p,r}$ , by requiring that under  $\mathbb{P}_{p,r}$  the skeleton is distributed according to  $\mathbf{P}_{p,r}$  and, conditionally on the skeleton, the triangulations with a boundary filling in the slots are independent and Boltzmann distributed (with boundary lengths prescribed by the skeleton). We can then restate Lemma 2 by saying that, if  $\Delta \in \mathbb{C}_{p,r}$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(p)}) = \Delta) = \mathbb{P}_{p,r}(\Delta), \quad (21)$$

or, in other words, the law of  $B_r^\bullet(\overline{\mathcal{T}}_n^{(p)})$  converges weakly to  $\mathbb{P}_{p,r}$  as  $n \rightarrow \infty$ .

For  $1 \leq r < r'$ , there is a canonical projection  $\mathbf{p}_{r,r'}$  from  $\mathbb{C}_{p,r'}$  onto  $\mathbb{C}_{p,r}$ , which maps  $\Delta \in \mathbb{C}_{p,r'}$  to  $B_r^\bullet(\Delta)$ . The preceding convergence then implies that the probability measures  $\mathbb{P}_{p,r}$ ,  $r \geq 1$ , are consistent in the sense that  $\mathbb{P}_{p,r} = \mathbb{P}_{p,r'} \circ \mathbf{p}_{r,r'}^{-1}$  for every  $1 \leq r < r'$ . It follows that we can define a random infinite triangulation of the plane, which we denote by  $\mathcal{T}_\infty^{(p)}$ , with a (simple) boundary of length  $p$ , such that  $\mathcal{T}_\infty^{(p)}$  has a unique end, and the law of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  is equal to  $\mathbb{P}_{p,r}$  for every integer  $r \geq 1$ . Note that here infinity plays the role of the distinguished point, so that the hull  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  is obtained by “filling in” the finite holes in the ball  $B_r(\mathcal{T}_\infty^{(p)})$ , which is itself the union of all faces that are incident to a vertex at graph distance strictly less than  $r$  from the boundary.

We call  $\mathcal{T}_\infty^{(p)}$  the (type I) uniform infinite triangulation of the  $p$ -gon. It follows from (21) that  $\mathcal{T}_\infty^{(p)}$  is the local limit of  $\mathcal{T}_n^{(p)}$  as  $n \rightarrow \infty$ . When  $p = 1$ , the boundary is a loop, and, after performing the inverse of the transformation described in Fig. 2, we get a random infinite planar triangulation, which we denote by  $\mathcal{T}_\infty$ . Then  $\mathcal{T}_\infty$  is the local limit of uniform rooted (plane) triangulations with  $n$  vertices when  $n \rightarrow \infty$ , and therefore is identified with the type I uniform infinite planar triangulation (UIPT), which was already constructed in [36, Proposition 6.2]. Notice that this approach via the skeleton decomposition gives a simple method for constructing the type I UIPT. A similar method was already used by Krikun [23] to construct the Uniform Infinite Planar Quadrangulation.

## 2.5 The comparison principle

Our goal in this section is to obtain a comparison principle showing that the law of the skeleton of the hull  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  is “not too different” from the law of a finite sequence of independent Galton–Watson trees.



We write  $L_r^{(p)}$  for the length of the top cycle of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$ . When  $p = 1$ , we will write  $L_r = L_r^{(1)}$  to simplify notation.

We first discuss a spatial Markov property of the law of  $\mathcal{T}_\infty^{(p)}$ . To this end, let  $1 \leq r < s$  be integers, and let  $\Delta \in \mathbb{C}_{p,s}$ . Let  $q$  be the length of the cycle  $\partial_r \Delta$ . Then  $\Delta$  is obtained by gluing a triangulation  $\Delta'' \in \mathbb{C}_{q,s-r}$  on the top cycle of another triangulation  $\Delta' \in \mathbb{C}_{p,r}$ , whose top cycle has length  $q$ . From the explicit formula for the probability measure  $\mathbf{P}_{p,r}$  on  $\mathbb{F}_{p,r}$ , it is a simple matter to verify that

$$\mathbb{P}_{p,s}(\Delta) = \mathbb{P}_{p,r}(\Delta') \times \mathbb{P}_{q,s-r}(\Delta'').$$

From this equality, it follows that, conditionally on  $\{L_r^{(p)} = q\}$ ,  $B_s^\bullet(\mathcal{T}_\infty^{(p)}) \setminus B_r^\bullet(\mathcal{T}_\infty^{(p)})$  is distributed according to  $\mathbb{P}_{q,s-r}$  and is independent of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  — we slightly abuse notation by writing  $B_s^\bullet(\mathcal{T}_\infty^{(p)}) \setminus B_r^\bullet(\mathcal{T}_\infty^{(p)})$  for the triangulation of the cylinder of height  $s - r$  consisting of the faces of  $\mathcal{T}_\infty^{(p)}$  that lie in  $B_s^\bullet(\mathcal{T}_\infty^{(p)})$  but not in  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$ , and this triangulation is rooted at the edge of  $\partial^* B_r^\bullet(\mathcal{T}_\infty^{(p)})$  that is the root of the first tree of the skeleton of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$ . By letting  $s \rightarrow \infty$ , we also get that, conditionally on  $\{L_r^{(p)} = q\}$ , the triangulation  $\mathcal{T}_\infty^{(p)} \setminus B_r^\bullet(\mathcal{T}_\infty^{(p)})$  is distributed as  $\mathcal{T}_\infty^{(q)}$  and is independent of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$ .

The following lemma provides useful estimates about the distribution of  $L_r = L_r^{(1)}$ .

**Lemma 4.** *There exists a constant  $C_0 > 0$  such that for any integer  $\alpha \geq 0$  and any choice of the integers  $r, p \geq 1$  we have*

$$\mathbb{P}(L_r = p) \leq \frac{C_0}{r^2}. \quad (22)$$

and

$$\mathbb{P}(L_r > \alpha r^2) \leq C_0 e^{-\alpha/5}, \quad (23)$$

The proof of this lemma is postponed to the end of the section.

We now fix a positive constant  $a \in (0, 1)$ , which may be chosen arbitrarily small. For every integer  $r \geq 1$ , we let  $N_r^{(a)}$  be uniformly distributed over  $\{\lfloor ar^2 \rfloor + 1, \dots, \lfloor a^{-1}r^2 \rfloor\}$ . We also consider a sequence  $\tau_1, \tau_2, \dots$  of independent Galton–Watson trees with offspring distribution  $\theta$ , which is independent of  $N_r^{(a)}$ . For every  $j \geq 0$ , we write  $[\tau_i]_j$  for the tree  $\tau_i$  truncated at generation  $j$  (that is, we remove all vertices at height strictly greater than  $j$ ).

If  $1 \leq r < s$ , we let  $\mathcal{F}_{r,s}^{(1)}$  be the skeleton of  $B_s^\bullet(\mathcal{T}_\infty^{(1)}) \setminus B_r^\bullet(\mathcal{T}_\infty^{(1)})$ . We also write  $\tilde{\mathcal{F}}_{r,s}^{(1)}$  for the (non-pointed) forest obtained by a random cyclic permutation of the trees of  $\mathcal{F}_{r,s}^{(1)}$  (so that the first tree in  $\tilde{\mathcal{F}}_{r,s}^{(1)}$  is the tree of index  $K$  in  $\mathcal{F}_{r,s}^{(1)}$ , where  $K$  is chosen uniformly over  $\{1, 2, \dots, L_s\}$ ) and also “forgetting” the distinguished vertex at generation  $s - r$ . On the event  $\{L_r = p\} \cap \{L_s = q\}$ ,  $\tilde{\mathcal{F}}_{r,s}^{(1)}$  is a random element of the set  $\mathbb{F}_{p,q,s-r}''$  introduced before Lemma 3.

**Proposition 5.** *There exists a constant  $C_1$ , which only depends on the real  $a$ , such that, for every sufficiently large integer  $r$ , for every choice of  $s \in \{r + 1, r + 2, \dots\}$ , for every choice of the integers  $p$  and  $q$  with  $ar^2 < p \leq a^{-1}r^2$  and  $ar^2 < q \leq a^{-1}r^2$ , for every forest  $\mathcal{F} \in \mathbb{F}_{p,q,s-r}''$ , we have*

$$\mathbb{P}(\tilde{\mathcal{F}}_{r,s}^{(1)} = \mathcal{F}) \leq C_1 \mathbb{P}([\tau_1]_{s-r}, \dots, [\tau_{N_r^{(a)}}]_{s-r} = \mathcal{F}). \quad (24)$$

*Proof.* By a standard formula for Galton–Watson trees, we have first

$$\begin{aligned} \mathbb{P}([\tau_1]_{s-r}, \dots, [\tau_{N_r^{(a)}}]_{s-r} = \mathcal{F}) &= \mathbb{P}(N_r^{(a)} = p) \mathbb{P}([\tau_1]_{s-r}, \dots, [\tau_p]_{s-r} = \mathcal{F}) \\ &= \frac{1}{\lfloor a^{-1}r^2 \rfloor - \lfloor ar^2 \rfloor} \times \prod_{v \in \mathcal{F}^*} \theta(c_v), \end{aligned} \quad (25)$$

using again the notation  $\mathcal{F}^*$  for the set of all vertices of  $\mathcal{F}$  at height strictly less than  $s - r$ . On the other hand, let  $\mathcal{F}^\circ$  be any (pointed) forest in  $\mathbb{F}_{p,q,s-r}$  such that the sequence of trees in  $\mathcal{F}^\circ$  coincides with that in  $\mathcal{F}$  up to a cyclic permutation. By the observations of the beginning of this section, we know that, conditionally on  $L_r = p$ ,  $B_s^\bullet(\mathcal{T}_\infty^{(1)}) \setminus B_r^\bullet(\mathcal{T}_\infty^{(1)})$  is distributed according to  $\mathbb{P}_{p,s-r}$ , and thus

$$\mathbb{P}(\mathcal{F}_{r,s}^{(1)} = \mathcal{F}^\circ \mid L_r = p) = \mathbf{P}_{p,s-r}(\mathcal{F}^\circ) = \frac{12^{-q}C(q)}{12^{-p}C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v),$$

Note that the right-hand side of the previous display only depends on  $\mathcal{F}$  and not on the choice of  $\mathcal{F}^\circ$ . By arguments similar to those of the proof of Lemma 3, we have then

$$\mathbb{P}(\mathcal{F}_{r,s}^{(1)} = \mathcal{F} \mid L_r = p) = \frac{p}{q} \mathbb{P}(\mathcal{F}_{r,s}^{(1)} = \mathcal{F}^\circ \mid L_r = p) = \frac{h(q)}{h(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v).$$

By our conditions on  $p$  and  $q$ , the ratio  $h(q)/h(p)$  is bounded above by a constant  $C_2$  (depending on  $a$ ). Using the bound (22), we thus get

$$\mathbb{P}(\tilde{\mathcal{F}}_{r,s}^{(1)} = \mathcal{F}) \leq \frac{C_0}{r^2} C_2 \prod_{v \in \mathcal{F}^*} \theta(c_v).$$

The bound of the proposition now follows by comparing the right-hand side of the previous display with the right-hand side of (25).  $\square$

*Proof of Lemma 4.* We observe that

$$\mathbb{P}(L_r = p) = \sum_{\mathcal{F} \in \mathbb{F}'_{1,p,r}} \mathbb{P}(\tilde{\mathcal{F}}_{0,r}^{(1)} = \mathcal{F}) = \sum_{\mathcal{F} \in \mathbb{F}'_{1,p,r}} \frac{h(p)}{h(1)} \prod_{v \in \mathcal{F}^*} \theta(c_v),$$

where  $\tilde{\mathcal{F}}_{0,r}^{(1)}$  is defined as above from the skeleton  $\mathcal{F}_{0,r}^{(1)}$  of the triangulation of the cylinder  $B_r^\bullet(\mathcal{T}_\infty^{(1)})$ . Hence

$$\mathbb{P}(L_r = p) = \frac{h(p)}{h(1)} \mathbb{P}_p(Y_r = 1) \quad (26)$$

where, as previously,  $(Y_n)_{n \geq 0}$  stands for a Galton–Watson process with offspring distribution  $\theta$  that starts from  $k$  under the probability measure  $\mathbb{P}_k$ . From the explicit form of  $h$  there exists a constant  $C_3$  such that

$$h(p) \leq \frac{C_3}{\sqrt{p}}, \quad (27)$$

for every  $p \geq 1$ . On the other hand, from (16), we have

$$\mathbb{P}_1(Y_r = 0) = 1 - (r+1)^{-2}. \quad (28)$$

and

$$\begin{aligned} \mathbb{P}_p(Y_r = 1) &= \lim_{x \downarrow 0} x^{-1} \left( \mathbb{E}_p[x^{Y_r}] - \mathbb{P}_p(Y_r = 0) \right) \\ &= \lim_{x \downarrow 0} x^{-1} \left( \left( 1 - \left( r + \frac{1}{\sqrt{1-x}} \right)^{-2} \right)^p - \left( 1 - (r+1)^{-2} \right)^p \right) \\ &= \frac{p}{(r+1)^3} (1 - (r+1)^{-2})^{p-1}. \end{aligned} \quad (29)$$

It follows that, with some constants  $C_4, C_5 > 0$ ,

$$\mathbb{P}(L_r = p) \leq \frac{C_3}{h(1)} \frac{\sqrt{p}}{(r+1)^3} (1 - (r+1)^{-2})^{p-1} \leq \frac{C_4}{r^2} \sqrt{\frac{p}{r^2}} e^{-(p-1)/(r+1)^2} \leq \frac{C_5}{r^2} \sqrt{\frac{p}{r^2}} e^{-p/(4r^2)}.$$

The bound (22) immediately follows. As for (23), we use the fact that the function  $x \mapsto \sqrt{x}e^{-x/4}$  is decreasing when  $x \geq 2$  so that, if  $\alpha \geq 2$ , we have, with some constant  $C_6$ ,

$$\mathbb{P}(L_r > \alpha r^2) \leq \sum_{q=\alpha r^2+1}^{\infty} \frac{C_5}{r^2} \sqrt{\frac{q}{r^2}} e^{-q/(4r^2)} \leq \frac{C_5}{r^2} \int_{\alpha r^2}^{\infty} dx \sqrt{\frac{x}{r^2}} e^{-x/(4r^2)} \leq C_6 e^{-\alpha/5}.$$

□

**Remark 3.** *The preceding calculations give for every  $x > 0$ ,*

$$\mathbb{P}_{\lfloor xr^2 \rfloor}(Y_r = 1) \underset{r \rightarrow \infty}{\sim} \frac{x}{r} \exp(-x),$$

*and, noting that  $\frac{h(p)}{h(1)} \sim 2/\sqrt{\pi p}$  as  $p \rightarrow \infty$ , we obtain that, for every  $x > 0$ ,*

$$\lim_{r \rightarrow \infty} r^2 \mathbb{P}(L_r = \lfloor xr^2 \rfloor) = \frac{2}{\sqrt{\pi}} \sqrt{x} \exp(-x).$$

*In this way we recover (in a stronger form) the fact that  $r^{-2}L_r$  converges in distribution to a Gamma distribution with parameter  $3/2$  [17, Theorem 2].*

We can apply Proposition 5 to get information on the probability of coalescence of left-most geodesics from distinct vertices of  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$  when  $n$  is large. Let  $u_0^{(n)}$  be chosen uniformly at random over the vertices of  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$ , and enumerate all vertices of  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$  in clockwise order as  $u_0^{(n)}, u_1^{(n)}, \dots, u_{L_n-1}^{(n)}$ . Fix  $\delta > 0$ . We claim that, if  $\eta \in (0, 1/2)$  is chosen small enough, the probability of the intersection of

$$\{an^2 \leq L_n \leq a^{-1}n^2\} \cap \{an^2 \leq L_{n-\lfloor \eta n \rfloor} \leq a^{-1}n^2\} \quad (30)$$

with the event where the left-most geodesics from  $u_0^{(n)}$  and  $u_{\lfloor an^2/2 \rfloor}^{(n)}$  coalesce before hitting  $\partial^* B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})$  is bounded above by  $\delta$  for all large enough  $n$ . Indeed, using the remarks at the end of subsection 2.2, we need to bound the probability of (the intersection of the event in (30) with) the event where the height of the subforest consisting of the first  $\lfloor an^2/2 \rfloor$  trees of  $\tilde{\mathcal{F}}_{n-\lfloor \eta n \rfloor, n}^{(1)}$  is strictly smaller than  $\lfloor \eta n \rfloor$ , or the same holds for the height of the complementary subforest. Proposition 5 shows that, up to a multiplicative constant depending only on  $a$ , this probability is bounded above by twice the probability that a forest of  $\lfloor an^2/2 \rfloor$  independent Galton–Watson trees with offspring distribution  $\theta$  has height smaller than  $\lfloor \eta n \rfloor$ , and our claim now follows from (28).

### 3 Half-plane models

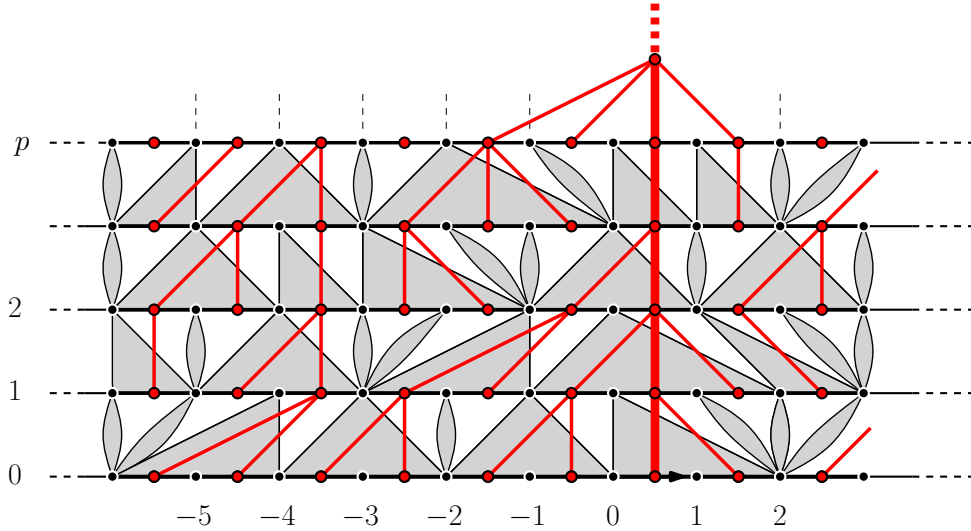
In this section, we introduce the two half-plane models that are local limits of large random triangulations of the cylinder rooted either on the bottom cycle (as previously) or on the top cycle. In the first case, we get the upper half-plane model, which was already discussed for type II triangulations in [5] and, in the second case, we get the lower half-plane model. We then obtain a relation between these two half-plane models (Proposition 8). The lower half-plane model is most relevant for our study — the time constants  $\mathbf{c}_0, \mathbf{c}_1$  and  $\mathbf{c}_2$  of the Introduction will arise in an application of the ergodic subadditive theorem on this random lattice (Propositions 18 and 25). However some of the delicate estimates that we will need about the geometry of the lower half-plane model are easier to derive first for the upper half-plane model and can then be transferred to the lower half-plane model using Proposition 8 and Corollary 9.

### 3.1 The upper half-plane triangulation

We construct a triangulation of the upper half-plane  $\mathbb{R} \times \mathbb{R}_+$ , whose vertex set contains all points of the form  $(i, j)$  for  $i \in \mathbb{Z}$  and  $j \in \mathbb{Z}_+$ . A key ingredient of this construction is an infinite tree, which is closely related to the Galton–Watson tree with offspring distribution  $\theta$  conditioned on non-extinction. This tree will be embedded in the half-plane so that its vertices are exactly all points of the form  $(\frac{1}{2} + i, j)$  for  $i \in \mathbb{Z}$  and  $j \in \mathbb{Z}_+$ . Let us start by describing this tree and its embedding.

The tree has an infinite *spine* which consists of all vertices of the discrete half-line  $\{(\frac{1}{2}, j), j \in \mathbb{Z}_+\}$ , with an edge between any two successive vertices on this half-line. Informally, one may think that time runs backwards when we move upward the spine, so that the vertex  $(\frac{1}{2}, j)$  is the parent of the vertex  $(\frac{1}{2}, j-1)$  for every  $j \geq 1$ . Write  $\bar{\theta}$  for the size-biased distribution associated with  $\theta$ , namely  $\bar{\theta}(k) = k\theta(k)$  for every  $k \geq 1$ . Every vertex of the form  $(\frac{1}{2}, j)$ ,  $j \geq 1$  has (independently of the others) a random number  $m_j$  of children distributed according to  $\bar{\theta}$ , and these are the vertices  $(\frac{1}{2} + k, j-1)$  for  $\ell_j - m_j \leq k \leq \ell_j - 1$ , where  $\ell_j$  is uniform over  $\{1, 2, \dots, m_j\}$  (put differently, the rank of  $(\frac{1}{2}, j-1)$  among the children of  $(\frac{1}{2}, j)$  is uniform). Of course the pairs  $(\ell_j, m_j)$ ,  $j \in \mathbb{Z}_+$ , are assumed to be independent.

Then, to every vertex of the form  $(\frac{1}{2} + k, j)$  ( $k \neq 0$ ) that is a child of a vertex of the spine, we attach (independently and independently of  $(\ell_i, m_i)_{i \in \mathbb{Z}_+}$ ) a Galton–Watson tree with offspring distribution  $\theta$  truncated at height  $j$ , in such a way that vertices at height  $r \in \{0, 1, \dots, j\}$  in this truncated tree will be points of the form  $(\frac{1}{2} + i, j-r)$ . An easy calculation shows that on both sides of the spine infinitely many of these trees will hit the maximal possible height. It follows that we may draw these trees in the upper half-plane, in such a way that edges do not cross and vertices (including those of the spine) are exactly all points of the form  $(\frac{1}{2} + i, j)$  for  $i \in \mathbb{Z}$  and  $j \in \mathbb{Z}_+$ . In particular, every vertex  $(\frac{1}{2} + i, j)$  with  $i \neq 0$  is a descendant of some vertex of the spine. Furthermore this embedding is unique. Rather than giving a more formal construction, we refer the reader to Fig. 5 from which the definition of our infinite tree should be clear.



**Figure 5:** Illustration of the construction of the upper half-plane triangulation. In red, the underlying tree giving the “skeleton” of the map and in grey, the slots to be filled in by independent Boltzmann triangulations. The thick red line represents the spine of the infinite tree.

Let us now construct our infinite triangulation of the upper half-plane. We start by constructing

special triangles, which we call the downward triangles by analogy with the previous sections, whose vertices will be elements of  $\mathbb{Z} \times \mathbb{Z}_+$ . We first require that, for every  $(i, j) \in \mathbb{Z} \times \mathbb{Z}_+$ , the horizontal edge  $[i, i+1] \times \{j\}$  connecting  $(i, j)$  to  $(i+1, j)$  is an edge of the triangulation. For every such horizontal edge with  $j \geq 1$ , we construct a *downward triangle* containing this edge, whose third vertex is the vertex  $(k, j-1)$ , where  $k$  is the minimal integer such that the tree vertex  $(\frac{1}{2} + k, j-1)$  is a child of  $(\frac{1}{2} + i', j)$  for some  $i' > i$ . We do this construction in such a way that edges are all distinct (and of course do not cross). Note in particular that if a (tree) vertex  $(\frac{1}{2} + i, j)$  with  $j \geq 1$  has no child, there will be a double edge from  $(i, j)$  to  $(k, j-1)$  for some  $k \in \mathbb{Z}$ . The configuration of downward triangles is then completely determined by the infinite tree. As in the previous sections, the complement of the union of downward triangles in the half-plane consists of slots, each vertex of the form  $(i, j)$ ,  $j \geq 1$ , being at the “top” of a slot bounded by a cycle whose length is 2 plus the number of children of  $(\frac{1}{2} + i, j)$  in the infinite tree. We then fill in the slots by independent Boltzmann triangulations with the prescribed perimeters, and, as previously, we make the convention that if a slot with perimeter 2 is filled in by the edge-triangulation, this means that the double edge bounding this slot is glued into a single edge. The resulting triangulation of the upper half-plane is called the UHPT for Upper Half-Plane Triangulation. It is rooted at the edge between  $(0, 0)$  and  $(1, 0)$ , which is oriented from left to right. We write  $\mathcal{U}$  for the UHPT and  $\partial\mathcal{U}$  for its (bottom) boundary.

**Proposition 6.** *We have*

$$\mathcal{T}_\infty^{(p)} \xrightarrow[p \rightarrow \infty]{(d)} \mathcal{U},$$

*in the sense of local limits of rooted planar maps.*

**Remark 4.** In [5], Angel uses a similar local convergence to define the type II Uniform Infinite Half Planar Triangulation. Our approach is however different from the peeling construction given in [5, 7].

*Proof.* We first observe that, for every fixed  $r \geq 1$  and  $j \geq 1$ ,

$$\mathbb{P}(L_r^{(p)} = j) \xrightarrow[p \rightarrow \infty]{} 0. \quad (31)$$

Indeed, the same arguments as in the proof of Lemma 4 give

$$\mathbb{P}(L_r^{(p)} = j) = \frac{h(j)}{h(p)} \mathbb{P}_j(Y_r = p) \leq \frac{h(j)}{h(p)} \frac{1}{p} \mathbb{E}_j[Y_r] = \frac{j h(j)}{p h(p)},$$

yielding the desired result since  $p h(p) \rightarrow \infty$  as  $p \rightarrow \infty$ .

In order to prove the proposition, it is enough to prove that, for every  $r \geq 1$ , if  $\mathcal{B}_r(\mathcal{U})$ , respectively  $\mathcal{B}_r(\mathcal{T}_\infty^{(p)})$ , denotes the planar map obtained by keeping only the faces of  $\mathcal{U}$ , resp. of  $\mathcal{T}_\infty^{(p)}$ , that are incident to a vertex at graph distance strictly less than  $r$  from the root vertex, we have

$$\mathbb{P}(\mathcal{B}_r(\mathcal{T}_\infty^{(p)}) = A) \xrightarrow[p \rightarrow \infty]{} \mathbb{P}(\mathcal{B}_r(\mathcal{U}) = A), \quad (32)$$

for any rooted planar map  $A$ . To get this convergence, fix  $r \geq 1$  and write  $\mathcal{F}_{0,r}^{(p)} = (\mathcal{T}_0^{(p)}, \mathcal{T}_1^{(p)}, \dots, \mathcal{T}_{L_r^{(p)}-1}^{(p)})$  for the skeleton of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$ . We will prove that, for every  $k \geq 1$ , if  $(\tau_{-k}, \dots, \tau_0, \dots, \tau_k)$  is a finite collection of plane trees having maximal height  $r$  and a distinguished vertex at height  $r$  that belongs to  $\tau_0$ ,

$$\begin{aligned} & \mathbb{P}(\{\mathcal{T}_{L_r^{(p)}-k}^{(p)} = \tau_{-k}, \dots, \mathcal{T}_{L_r^{(p)}-1}^{(p)} = \tau_{-1}, \mathcal{T}_0^{(p)} = \tau_0, \mathcal{T}_1^{(p)} = \tau_1, \dots, \mathcal{T}_k^{(p)} = \tau_k\} \cap \{L_r^{(p)} \geq 2k+1\}) \\ & \xrightarrow[p \rightarrow \infty]{} \mathbb{P}(\Gamma_{(-k,r)} = \tau_{-k}, \dots, \Gamma_{(0,r)} = \tau_0, \dots, \Gamma_{(k,r)} = \tau_k) \end{aligned} \quad (33)$$

where  $\Gamma_{(i,j)}$  stands for the subtree of descendants of  $(\frac{1}{2}+i, j)$  in the infinite tree (here we view  $\Gamma_{(i,j)}$  as an abstract plane tree, and we “forget” the embedding in the plane), and it is understood that  $\Gamma_{(0,r)}$  has a distinguished vertex corresponding to  $(\frac{1}{2}, 0)$ , so that when we write the equalities  $\mathcal{T}_0^{(p)} = \tau_0$  or  $\Gamma_{(0,r)} = \tau_0$ , we mean an equality of pointed trees. We observe that, if  $k$  is large, we can find a collection  $\mathbf{F}_k$  of forests  $(\tau_{-k}, \dots, \tau_0, \dots, \tau_k)$  such that the probability of the event  $\{(\Gamma_{(-k,r)}, \dots, \Gamma_{(k,r)}) \in \mathbf{F}_k\}$  is close to 1, and, on the latter event, the ball  $\mathcal{B}_r(\mathcal{U})$  is a deterministic function of the trees  $\Gamma_{(-k,r)}, \dots, \Gamma_{(k,r)}$  and of the triangulations with a boundary filling in the slots associated with the vertices of these trees. Similarly, on the event  $\{(\mathcal{T}_{L_r^{(p)}-k}^{(p)}, \dots, \mathcal{T}_{L_r^{(p)}-1}^{(p)}, \mathcal{T}_0^{(p)}, \dots, \mathcal{T}_k^{(p)}) \in \mathbf{F}_k\} \cap \{L_r^{(p)} \geq 2k+1\}$ , the ball  $\mathcal{B}_r(\mathcal{T}_\infty^{(p)})$  will be the same deterministic function of the trees  $\mathcal{T}_{L_r^{(p)}-k}^{(p)}, \dots, \mathcal{T}_{L_r^{(p)}-1}^{(p)}, \mathcal{T}_0^{(p)}, \dots, \mathcal{T}_k^{(p)}$  and of the associated triangulations with a boundary. The desired convergence (32) thus follows from (33), using also (31).

It remains to prove (33) and, to this end, we fix a forest  $(\tau_{-k}, \dots, \tau_k)$  satisfying the assumptions stated above. We first note that, if  $\mathbf{V}^*(\tau_i)$  stands for the collection of all vertices of  $\tau_i$  at height strictly less than  $r$ , we have

$$\mathbb{P}(\Gamma_{(-k,r)} = \tau_{-k}, \dots, \Gamma_{(0,r)} = \tau_0, \dots, \Gamma_{(k,r)} = \tau_k) = \prod_{v \in \mathbf{V}^*(\tau_{-k}) \cup \dots \cup \mathbf{V}^*(\tau_k)} \theta(c_v), \quad (34)$$

where  $c_v$  denotes the number of children of  $v$ . The preceding equality holds because by construction the trees  $\Gamma_{(i,r)}$ ,  $i \neq 0$  are independent Galton–Watson trees with offspring distribution  $\theta$  truncated at height  $r$ , and the tree  $\Gamma_{(0,r)}$  is a size-biased Galton–Watson tree with offspring distribution  $\theta$  truncated at height  $r$  and given with a distinguished vertex at height  $r$ . See [29] for the definition and properties of size-biased Galton–Watson trees, noting that, if we “forget” the distinguished vertex, the right-hand side of the preceding formula has an extra multiplicative factor equal to the size of generation  $r$  in  $\tau_0$ .

Consider then the left-hand side of (33). To simplify notation, write  $\mathcal{F}_{(k)}$  for the forest  $(\tau_{-k}, \dots, \tau_k)$  and  $m_k$  for the number of vertices of  $\mathcal{F}_{(k)}$  at generation  $r$ . For any forest  $\mathcal{F} = (\sigma_0, \dots, \sigma_\ell) \in \mathbb{F}_{p,\ell,r}$ , with  $\ell \geq 2k+1$ , write  $\Phi_k(\mathcal{F}) = (\sigma_{\ell-k}, \dots, \sigma_{\ell-1}, \sigma_0, \dots, \sigma_k)$  where it is understood that, in  $\Phi_k(\mathcal{F})$  as in  $\mathcal{F}$ ,  $\sigma_0$  comes with a distinguished vertex at height  $r$ . Then, using (20) and the fact that the law of  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  is  $\mathbf{P}_{p,r}$ , we can rewrite the left-hand side of (33) as

$$\begin{aligned} & \sum_{\ell=2k+1}^{\infty} \sum_{\mathcal{F} \in \mathbb{F}_{p,\ell,r} : \Phi_k(\mathcal{F}) = \mathcal{F}_{(k)}} \frac{12^{-\ell} C(\ell)}{12^{-p} C(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v) = \left( \prod_{v \in \mathbf{V}^*(\tau_{-k}) \cup \dots \cup \mathbf{V}^*(\tau_k)} \theta(c_v) \right) \\ & \times \left( \sum_{\ell=2k+1}^{\infty} \frac{12^{-\ell} C(\ell)}{12^{-p} C(p)} \sum_{\substack{\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_{\ell-k-1} \\ \# \sigma_{k+1}(r) + \dots + \# \sigma_{\ell-k-1}(r) = p - m_k}} \prod_{v \in \mathbf{V}^*(\sigma_{k+1}) \cup \dots \cup \mathbf{V}^*(\sigma_{\ell-k-1})} \theta(c_v) \right), \end{aligned} \quad (35)$$

where the second sum in the last line is over all choices of the plane trees  $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_{\ell-k-1}$  having a total number of vertices at height  $r$  equal to  $p - m_k$ . Set  $\varphi(\ell) = 12^{-\ell} C(\ell)$  to simplify notation, and write  $A_p$  for the quantity inside parentheses in the second line of (35). Then we have

$$\begin{aligned} A_p &= \sum_{\ell=2k+1}^{\infty} \frac{\varphi(\ell)}{\varphi(p)} \mathbb{P}_{\ell-(2k+1)}(Y_r = p - m_k) = \sum_{\ell=0}^{\infty} \frac{\varphi(\ell + 2k + 1)}{\varphi(p)} \mathbb{P}_\ell(Y_r = p - m_k) \\ &\geq \sum_{\ell=0}^{\infty} \frac{\varphi(\ell)}{\varphi(p)} \mathbb{P}_\ell(Y_r = p - m_k), \end{aligned}$$

since  $\varphi$  is monotone increasing. Fix  $\varepsilon \in (0, 1/2)$ . Recalling that  $\varphi(\ell) = \ell h(\ell)$ , we have then

$$A_p \geq (1 - \varepsilon) \sum_{\ell = \lfloor (1-\varepsilon)p \rfloor + 1}^{\infty} \frac{h(\ell)}{h(p)} \mathbb{P}_\ell(Y_r = p - m_k). \quad (36)$$

On the other hand, we claim that

$$\lim_{p \rightarrow \infty} \sum_{\ell=0}^{\lfloor (1-\varepsilon)p \rfloor} \frac{h(\ell)}{h(p)} \mathbb{P}_\ell(Y_r = p - m_k) = 0. \quad (37)$$

To see this, observe that the law of  $Y_r - \ell$  under  $\mathbb{P}_\ell$  is the law of the sum of  $\ell$  centered i.i.d. random variables whose tail asymptotics are given by (17). As a consequence of [19, Corollary 2.1], there exist constants  $C_\varepsilon$  and  $C'_\varepsilon$  such that, for every sufficiently large  $p$  and every  $\ell \in \{0, 1, \dots, \lfloor (1-\varepsilon)p \rfloor\}$ ,

$$\mathbb{P}_\ell(Y_r = p - m_k) \leq C_\varepsilon \ell \mathbb{P}_1(Y_r = p - \ell + 1 - m_k) \leq C'_\varepsilon \ell p^{-5/2},$$

using (17) in the last bound (to be precise, [19, Corollary 2.1] gives this only for “large” values  $\ell \geq \ell_0$  for some integer  $\ell_0$ , but the values  $\ell \leq \ell_0$  are easily treated by a direct argument). Since  $h(k) \sim 1/\sqrt{\pi k}$  as  $k \rightarrow \infty$ , we then get, for all sufficiently large  $p$ ,

$$\sum_{\ell=0}^{\lfloor (1-\varepsilon)p \rfloor} \frac{h(\ell)}{h(p)} \mathbb{P}_\ell(Y_r = p - m_k) \leq C''_\varepsilon p^{-2} \sum_{\ell=0}^{\lfloor (1-\varepsilon)p \rfloor} \ell^{1/2}$$

which tends to 0 as  $p \rightarrow \infty$ , proving our claim (37).

Using (36) and (37), we have then

$$\liminf_{p \rightarrow \infty} A_p \geq (1 - \varepsilon) \liminf_{p \rightarrow \infty} \sum_{\ell=0}^{\infty} \frac{h(\ell)}{h(p)} \mathbb{P}_\ell(Y_r = p - m_k) = (1 - \varepsilon) \liminf_{p \rightarrow \infty} \frac{h(p - m_k)}{h(p)},$$

by (19). Since  $h(p - m_k)/h(p)$  tends to 1 and  $\varepsilon$  was arbitrary, we have indeed proved that

$$\liminf_{p \rightarrow \infty} A_p \geq 1.$$

From (34) and (35), we get that the liminf of the quantities in the left-hand side of (33) is greater than or equal to the right-hand side, for any choice of the forest  $(\tau_{-k}, \dots, \tau_k)$ . On the other hand the sum of the quantities in the right-hand side over possible choices of  $(\tau_{-k}, \dots, \tau_k)$  is equal to 1. It follows that the convergence (33) holds. This completes the proof of the proposition.  $\square$

### 3.2 The lower half-plane triangulation

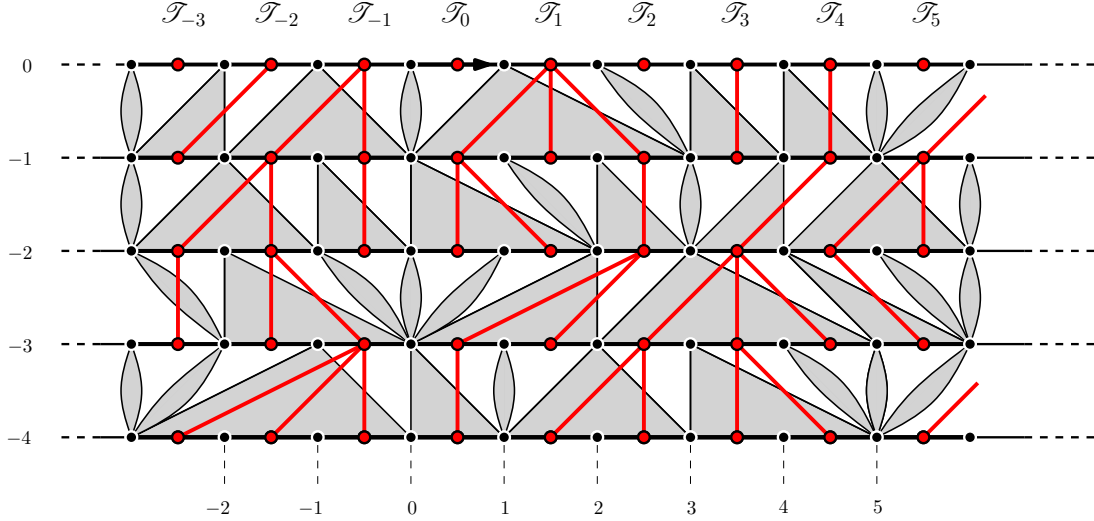
We now discuss the lower half-plane triangulation or LHPT, which can be obtained as the local limit in distribution of the hulls  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  when  $r \rightarrow \infty$ , provided that these hulls are re-rooted at an edge chosen uniformly on the top cycle.

The construction of the LHPT is similar to that of the UHPT in the previous section. The vertex set now contains all points of  $\mathbb{Z} \times \mathbb{Z}_-$ , and the role of the infinite tree is played by a doubly infinite sequence  $(\mathcal{T}_i)_{i \in \mathbb{Z}}$  of independent Galton–Watson trees with offspring distribution  $\theta$ . These trees are then embedded in the lower half-plane so that the root of  $\mathcal{T}_i$  is  $(\frac{1}{2} + i, 0)$  for every  $i \in \mathbb{Z}$ , and the collection of all vertices of the trees is exactly the set of all points of the form  $(\frac{1}{2} + i, j)$ , where  $(i, j) \in \mathbb{Z} \times \mathbb{Z}_-$  (vertices at height  $k$  in a tree being of the form  $(\frac{1}{2} + i, -k)$ ). Here there are several ways of doing this embedding, but for definiteness we may agree that the collection of vertices of the trees  $\mathcal{T}_i$  for  $i \geq 0$  is  $(\frac{1}{2} + \mathbb{Z}_+) \times \mathbb{Z}_-$ . See Fig. 6.

Given the trees, the downward triangles of the LHPT are constructed in a very similar way to what was done in the previous section. For every horizontal edge  $[i, i + 1] \times \{j\}$  connecting  $(i, j)$  to  $(i + 1, j)$ , where  $i \in \mathbb{Z}$  and  $j \in \mathbb{Z}_-$ , we construct a downward triangle containing this edge whose third vertex is the



vertex  $(k, j - 1)$ , where  $k$  is the minimal integer such that  $(\frac{1}{2} + k, j - 1)$  is a child of  $(\frac{1}{2} + i', j)$  for some  $i' > i$ . We then fill in the slots left by the downward triangles by independent Boltzmann triangulations with a boundary to get the LHPT, which is denoted by  $\mathcal{L}$ . By convention  $\mathcal{L}$  is rooted at the edge between  $(0, 0)$  and  $(1, 0)$ , which is oriented from left to right.



**Figure 6:** Illustration of the construction of the LHPT.

**Proposition 7.** *Let  $p \geq 1$ , and, for every  $r \geq 1$ , let  $\tilde{B}_r^\bullet(\mathcal{T}_\infty^{(p)})$  stand for the hull  $B_r^\bullet(\mathcal{T}_\infty^{(p)})$  re-rooted at an edge chosen uniformly at random on the top boundary and oriented so that the top face is lying on its left-hand side. Then,*

$$\tilde{B}_r^\bullet(\mathcal{T}_\infty^{(p)}) \xrightarrow[r \rightarrow \infty]{(d)} \mathcal{L},$$

*in the sense of local limits of rooted planar maps.*

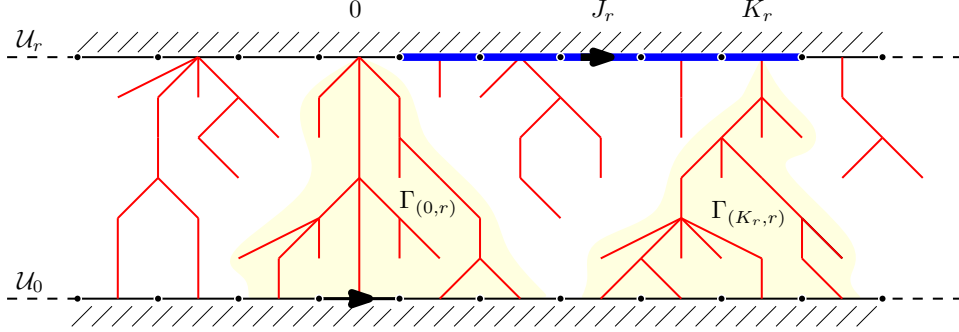
This can be proved by arguments very similar to the proof of Proposition 6. We omit the details as this statement is not needed in what follows.

Let us now discuss the connections between  $\mathcal{U}$  and  $\mathcal{L}$ . For every integer  $r \geq 1$ , we let  $\mathcal{U}_{[0,r]}$  stand for the (infinite) rooted planar map obtained by keeping only the first  $r$  layers of  $\mathcal{U}$ . More precisely, in our construction of the UHPT we keep only those vertices and edges that lie in the strip  $\mathbb{R} \times [0, r]$ . Alternatively, we may view  $\mathcal{U}_{[0,r]}$  as the hull of radius  $r$ , corresponding to distances from the bottom boundary. Similarly, we write  $\mathcal{L}_{[0,r]}$  for the rooted planar map obtained by keeping only the first  $r$  layers of  $\mathcal{L}$ , that is, the part of  $\mathcal{L}$  lying in the strip  $\mathbb{R} \times [-r, 0]$ . We will also use the notation  $\mathcal{U}_r$ , resp.  $\mathcal{L}_r$ , for the horizontal line  $\mathcal{U}_r = \{(i, r) : i \in \mathbb{Z}\}$ , resp.  $\mathcal{L}_r = \{(i, -r) : i \in \mathbb{Z}\}$ .

One may expect that the two infinite planar maps  $\mathcal{U}_{[0,r]}$  and  $\mathcal{L}_{[0,r]}$  are closely related, and that informally  $\mathcal{L}_{[0,r]}$  should correspond to  $\mathcal{U}_{[0,r]}$  re-rooted at an edge of its upper boundary. To give a precise statement, we introduce some notation.

We fix  $r \geq 1$  and recall our notation  $\Gamma_{(i,r)}$  for the subtree of descendants of  $(\frac{1}{2} + i, r)$  in the infinite tree associated with  $\mathcal{U}$ . We already noticed that the trees  $\Gamma_{(i,r)}$ ,  $i \neq 0$  are independent Galton–Watson trees with offspring distribution  $\theta$  truncated at height  $r$ , and the tree  $\Gamma_{(0,r)}$  is a size-biased Galton–Watson tree with offspring distribution  $\theta$  truncated at height  $r$  and given with a uniform distinguished vertex at

height  $r$ . For every  $i \in \mathbb{Z}$ , let  $\Gamma_{(i,r)}(r)$  stand for the set of vertices of  $\Gamma_{(i,r)}$  at height  $r$ . Also let  $K_r \geq 1$  be the first index  $i \geq 1$  such that  $\Gamma_{(i,r)}(r) \neq \emptyset$ . See Fig. 7.



**Figure 7:** From  $\mathcal{U}_{[0,r]}$  to  $\tilde{\mathcal{U}}_{[0,r]}$ . The thick part of the line  $\mathcal{U}_r$  corresponds to the possible edges at which the map can be re-rooted.

We also let  $i_r < 0$  be the largest integer  $i < 0$  such that  $\mathcal{T}_i$  has height at least  $r$ . As previously, let  $\mathcal{T}_{i_r}(r)$  be the set of all vertices of  $\mathcal{T}_{i_r}$  at height  $r$ .

**Proposition 8.** *Let  $\tilde{\mathcal{U}}_{[0,r]}$  stand for the infinite rooted planar map obtained by re-rooting  $\mathcal{U}_{[0,r]}$  so that the root edge is the horizontal edge from  $(J_r, r)$  to  $(J_r + 1, r)$ , where the index  $J_r$  is chosen uniformly at random in  $\{1, \dots, K_r\}$ . Then, for any nonnegative measurable function  $f$ ,*

$$\mathbb{E}[K_r f(\tilde{\mathcal{U}}_{[0,r]})] = \mathbb{E}[\#\mathcal{T}_{i_r}(r) f(\mathcal{L}_{[0,r]})].$$

*Proof.* Clearly, it is enough to verify that the distribution of the configuration of downward triangles is the same for  $\tilde{\mathcal{U}}_{[0,r]}$ , under the measure having density  $K_r$  with respect to  $\mathbb{P}$ , and for  $\mathcal{L}_{[0,r]}$ , under the measure having density  $\#\mathcal{T}_{i_r}(r)$  with respect to  $\mathbb{P}$ . In both cases, the configuration of downward triangles is coded by a doubly infinite sequence of trees, and we need to verify that these two sequences have the same distribution. By construction, the sequence of trees associated with  $\tilde{\mathcal{U}}_{[0,r]}$  is  $(\Gamma'_i)_{i \in \mathbb{Z}}$ , where  $\Gamma'_i = \Gamma_{(i+J_r,r)}$  if  $i \neq -J_r$ , and  $\Gamma'_{-J_r} = \Gamma_{(0,r)}^{\text{unp}}$ , where we use the notation  $\Gamma_{(0,r)}^{\text{unp}}$  to represent the tree  $\Gamma_{(0,r)}$  without its distinguished vertex at height  $r$ . Recall also that the trees  $(\mathcal{T}_i)_{i \in \mathbb{Z}}$  coding the configuration of downward triangles in  $\mathcal{L}$  are just independent Galton–Watson trees with offspring distribution  $\theta$ .

Fix four integers  $k_1, k_2, \ell_1, \ell_2$  such that  $0 < k_1 \leq \ell_1$  and  $0 \leq k_2 \leq \ell_2$ . Consider a finite sequence  $(\tau_{-\ell_1}, \tau_{-\ell_1+1}, \dots, \tau_{\ell_2})$  of  $\ell_1 + \ell_2 + 1$  trees having height less than or equal to  $r$  and such that the trees  $\tau_i$  for  $-k_1 < i < k_2$  have height strictly less than  $r$ , whereas  $\tau_{-k_1}$  and  $\tau_{k_2}$  have height  $r$ . By construction, we have

$$\{\Gamma'_{-\ell_1} = \tau_{-\ell_1}, \dots, \Gamma'_{\ell_2} = \tau_{\ell_2}\} = \{\Gamma_{(-\ell_1+k_1,r)} = \tau_{-\ell_1}, \dots, \Gamma_{(0,r)}^{\text{unp}} = \tau_{-k_1}, \dots, \Gamma_{(\ell_2+k_1,r)} = \tau_{\ell_2}\} \cap \{J_r = k_1\}.$$

Since we have  $K_r = k_1 + k_2$  on the first event in the right-hand side, it follows that

$$\begin{aligned} \mathbb{P}(\Gamma'_{-\ell_1} = \tau_{-\ell_1}, \dots, \Gamma'_{\ell_2} = \tau_{\ell_2}) &= \frac{1}{k_1 + k_2} \mathbb{P}(\Gamma_{(-\ell_1+k_1,r)} = \tau_{-\ell_1}, \dots, \Gamma_{(0,r)}^{\text{unp}} = \tau_{-k_1}, \dots, \Gamma_{(\ell_2+k_1,r)} = \tau_{\ell_2}) \\ &= \frac{\#\tau_{-k_1}(r)}{k_1 + k_2} \mathbb{P}([\mathcal{T}_{-\ell_1}]_r = \tau_{-\ell_1}, \dots, [\mathcal{T}_{\ell_2}]_r = \tau_{\ell_2}) \end{aligned}$$

where  $[\mathcal{T}_i]_r$  denotes the tree  $\mathcal{T}_i$  truncated at height  $r$ , and we used the fact that  $\Gamma_{(0,r)}^{\text{unp}}$  is a size-biased Galton–Watson tree with offspring distribution  $\theta$  truncated at height  $r$ . The statement of the proposition

follows since  $K_r = k_1 + k_2$  on  $\{\Gamma'_{-\ell_1} = \tau_{-\ell_1}, \dots, \Gamma'_{\ell_2} = \tau_{\ell_2}\}$  and  $i_r = -k_1$  on  $\{[\mathcal{T}_{-\ell_1}]_r = \tau_{-\ell_1}, \dots, [\mathcal{T}_{\ell_2}]_r = \tau_{\ell_2}\}$ .  $\square$

**Corollary 9.** *For every  $\varepsilon > 0$ , we can choose  $\delta > 0$  small enough, so that for every  $r \geq 1$ , for every measurable set  $A$ , the property  $\mathbb{P}(\tilde{\mathcal{U}}_{[0,r]} \in A) \leq \delta$  implies  $\mathbb{P}(\mathcal{L}_{[0,r]} \in A) \leq \varepsilon$ .*

*Proof.* Since  $\mathcal{T}_{i_r}$  is just a Galton–Watson tree with offspring distribution  $\theta$  conditioned on non-extinction at generation  $r$ , the generating function of  $\#\mathcal{T}_{i_r}(r)$  is derived from (16) and (28),

$$\mathbb{E}[x^{\#\mathcal{T}_{i_r}(r)}] = (r+1)^2 \left( (r+1)^{-2} - \left( r + \frac{1}{\sqrt{1-x}} \right)^{-2} \right), \quad x \in [0, 1].$$

From this, it is elementary to verify that  $(r+1)^{-2} \#\mathcal{T}_{i_r}(r)$  converges in distribution to a random variable  $U$  with Laplace transform  $\mathbb{E}[e^{-\lambda U}] = 1 - (1 + \lambda^{-1/2})^{-2}$ . Since  $U > 0$  a.s., we can find  $\eta > 0$  such that  $\mathbb{P}(\#\mathcal{T}_{i_r}(r) < \eta(r+1)^2) \leq \varepsilon/2$  for every  $r \geq 1$ .

We take  $\delta = \eta^2 \varepsilon^2 / 16$  and consider a measurable set  $A$  such that  $\mathbb{P}(\tilde{\mathcal{U}}_{[0,r]} \in A) \leq \delta$ . By Proposition 8, we have for every  $r \geq 1$ ,

$$\mathbb{E}[K_r \mathbf{1}_A(\tilde{\mathcal{U}}_{[0,r]})] = \mathbb{E}[\#\mathcal{T}_{i_r}(r) \mathbf{1}_A(\mathcal{L}_{[0,r]})], \quad (38)$$

and (28) shows that the distribution of  $K_r$  is given by  $\mathbb{P}(K_r \geq j) = (1 - (r+1)^{-2})^{j-1}$  for every  $j \geq 1$ . Straightforward calculations then give  $\mathbb{E}[(K_r)^2] \leq 4(r+1)^4$  and the Cauchy–Schwarz inequality implies that the left-hand side of (38) is bounded above by

$$\mathbb{E}[(K_r)^2]^{1/2} \mathbb{P}(\tilde{\mathcal{U}}_{[0,r]} \in A)^{1/2} \leq \frac{\eta \varepsilon}{2} (r+1)^2.$$

The right-hand side of (38) is bounded below by

$$\eta (r+1)^2 \mathbb{P}(\{\mathcal{L}_{[0,r]} \in A\} \cap \{\#\mathcal{T}_{i_r}(r) \geq \eta(r+1)^2\}).$$

By combining the preceding two bounds we get

$$\mathbb{P}(\{\mathcal{L}_{[0,r]} \in A\} \cap \{\#\mathcal{T}_{i_r}(r) \geq \eta(r+1)^2\}) \leq \frac{\varepsilon}{2}.$$

By our choice of  $\eta$ , we also know that  $\mathbb{P}(\#\mathcal{T}_{i_r}(r) < \eta(r+1)^2) \leq \varepsilon/2$ , and we get  $\mathbb{P}(\mathcal{L}_{[0,r]} \in A) \leq \varepsilon$ . This completes the proof.  $\square$

## 4 Estimates for distances along the boundary

In this section, we derive asymptotic estimates for distances on the boundary of the UHPT or of the LHPT, which roughly say that the graph distance (with respect to the half-plane triangulation) between boundary vertices grows like the square root of their distance along the boundary. We are interested in the case of the LHPT for future applications, but, for technical reasons, we start with the case of the UHPT. To derive these estimates in Section 4.2, we first study the layers of balls centered at the root vertex of the UHPT, in the spirit of Section 2.2.

### 4.1 Layers of balls in the UHPT

Let  $r \geq 1$  be an integer, which will be fixed throughout this subsection. The ball  $\mathcal{B}_r(\mathcal{U})$  is defined as the union of all triangles of  $\mathcal{U}$  which are incident to a vertex at graph distance smaller than or equal to  $r-1$  from the root vertex, and the hull  $\mathcal{B}_r^*(\mathcal{U})$  is the complement of the unique infinite component of

the complement of  $\mathcal{B}_r(\mathcal{U})$ . Then  $\mathcal{B}_r^\bullet(\mathcal{U})$  is a triangulation with a simple boundary consisting of (finitely many) edges on the boundary of  $\mathcal{U}$ , including the root edge, and a simple path formed by non-boundary edges of  $\mathcal{U}$  that connects the two extreme vertices of  $\mathcal{B}_r(\mathcal{U})$  lying on the boundary of  $\mathcal{U}$ . It will be useful to keep the information given by these two extreme vertices. So we view  $\mathcal{B}_r^\bullet(\mathcal{U})$  as a triangulation with a simple boundary, given with two distinguished vertices on the boundary, which are distinct and distinct from the root vertex.

**Lemma 10.** *Let  $A$  be a triangulation with a boundary, given with two distinct distinguished vertices on the boundary other than the root vertex. We write  $\tilde{\partial}A$  for the part of  $\partial A$  that consists of the path between the two distinguished vertices that contains the root edge. Assume that  $\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A) > 0$ . Let  $m \geq 2$  be the number of edges of  $\tilde{\partial}A$  and let  $q \geq 1$  be the number of edges of  $\partial A \setminus \tilde{\partial}A$ . Also let  $N \geq 0$  be the number of vertices of  $A$  that do not belong to  $\tilde{\partial}A$ . Then,*

$$\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A) = 12^{q-m} (12\sqrt{3})^{-N}.$$

*Proof.* If  $A'$  is another triangulation with a boundary, we use the notation  $A \sqsubset A'$  to mean that  $A$  can be obtained as a subtriangulation of  $A'$  with root edges coinciding and in such a way that  $\tilde{\partial}A$  is part of  $\partial A'$  and no other edge of  $A$  is on  $\partial A'$ . By Proposition 6, under the condition  $\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A) > 0$ ,

$$\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A) = \lim_{p \rightarrow \infty} \mathbb{P}(A \sqsubset \mathcal{T}_\infty^{(p)}).$$

On the other hand, the fact that  $\mathcal{T}_\infty^{(p)}$  is the local limit in distribution of the finite triangulations  $\mathcal{T}_n^{(p)}$  ensures that

$$\mathbb{P}(A \sqsubset \mathcal{T}_\infty^{(p)}) = \lim_{n \rightarrow \infty} \mathbb{P}(A \sqsubset \mathcal{T}_n^{(p)}).$$

Fix  $p > m$ , and note that the property  $A \sqsubset \mathcal{T}_n^{(p)}$  will hold if and only if  $\mathcal{T}_n^{(p)}$  is obtained by gluing a triangulation with a boundary of length  $q + (p - m)$  to  $A$ , in such a way that a part (of length  $q$ ) of the boundary of the glued triangulation is identified to  $\partial A \setminus \tilde{\partial}A$  (this should be made more precise by saying that the root edge of the glued triangulation is glued to a specific edge of  $\partial A \setminus \tilde{\partial}A$ ). This argument shows that, for  $n$  large enough,

$$\mathbb{P}(A \sqsubset \mathcal{T}_n^{(p)}) = \frac{\#\mathbb{T}_{n-N, p+q-m}}{\#\mathbb{T}_{n,p}}.$$

In a way similar to the derivation of (13), we now use (6) to get

$$\mathbb{P}(A \sqsubset \mathcal{T}_\infty^{(p)}) = \frac{C(p+q-m)}{C(p)} (12\sqrt{3})^{-N},$$

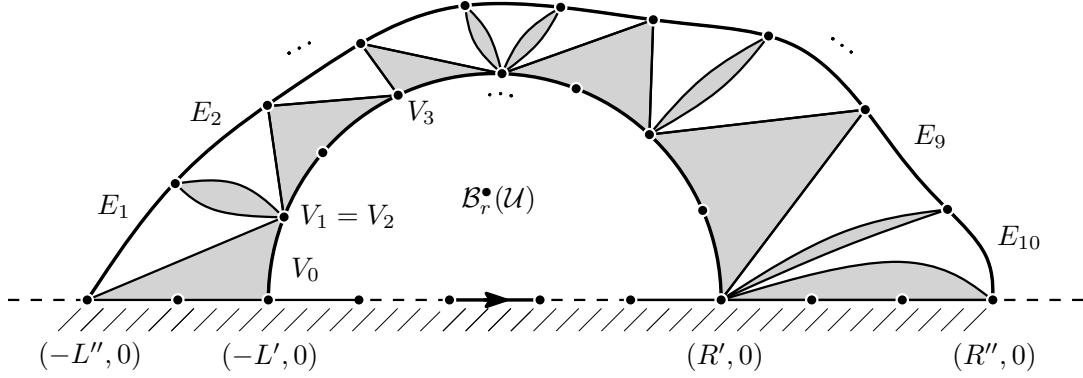
and then,

$$\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A) = \lim_{p \rightarrow \infty} \mathbb{P}(A \sqsubset \mathcal{T}_\infty^{(p)}) = 12^{q-m} (12\sqrt{3})^{-N},$$

which completes the proof.  $\square$

Our next goal is to describe the conditional distribution of the “layer”  $\mathcal{B}_{r+1}^\bullet(\mathcal{U}) \setminus \mathcal{B}_r^\bullet(\mathcal{U})$  given  $\mathcal{B}_r^\bullet(\mathcal{U})$ . We call internal edge of  $\partial \mathcal{B}_r^\bullet(\mathcal{U})$  any edge of  $\partial \mathcal{B}_r^\bullet(\mathcal{U})$  that does not belong to  $\partial \mathcal{U} = \mathcal{U}_0$ . We order the internal edges of  $\partial \mathcal{B}_{r+1}^\bullet(\mathcal{U})$  in clockwise order and denote them as  $E_1, E_2, \dots, E_Q$ , where  $Q \geq 1$  is the number of internal edges of  $\partial \mathcal{B}_{r+1}^\bullet(\mathcal{U})$ . We also let  $(-L', 0)$  be the left-most vertex of  $\partial \mathcal{B}_r^\bullet(\mathcal{U}) \cap \partial \mathcal{U}$ , and  $(R', 0)$  be the right-most vertex of  $\partial \mathcal{B}_r^\bullet(\mathcal{U}) \cap \partial \mathcal{U}$ . We define similarly  $L''$  and  $R''$  replacing  $\mathcal{B}_r^\bullet(\mathcal{U})$  by  $\mathcal{B}_{r+1}^\bullet(\mathcal{U})$ . See Fig. 8 for an illustration.

Any internal edge of  $\partial\mathcal{B}_{r+1}^\bullet(\mathcal{U})$  connects two vertices at distance  $r+1$  from the root vertex and is incident to a “downward” triangle whose third vertex belongs to  $\partial\mathcal{B}_r^\bullet(\mathcal{U})$ . Write  $V_1, \dots, V_Q$  for the vertices of  $\partial\mathcal{B}_r^\bullet(\mathcal{U})$  that belong to the downward triangles associated with  $E_1, \dots, E_Q$  respectively. Note that  $V_1, \dots, V_Q$  are not necessarily distinct. For  $1 \leq j \leq Q+1$ , write  $S_j$  for the number of edges of  $\partial\mathcal{B}_r^\bullet(\mathcal{U})$  lying between  $V_{j-1}$  and  $V_j$ , where by convention  $V_0$  is the vertex  $(-L', 0)$ , and  $V_{Q+1}$  is the vertex  $(R', 0)$ . Note that  $S_1 + \dots + S_{Q+1} =: \mathbf{P}_r$  is the number of internal edges of  $\partial\mathcal{B}_r^\bullet(\mathcal{U})$ .



**Figure 8:** Illustration of the setting of Proposition 11: the layer  $\mathcal{B}_{r+1}^\bullet(\mathcal{U}) \setminus \mathcal{B}_r^\bullet(\mathcal{U})$  is displayed. As usual the “slots” are represented in grey and not filled in for the clarity of the figure. Comparing with the skeleton decomposition of Section 2.2, we notice the particular roles played by the two extremes slots of the layer.

**Proposition 11.** *Let  $q \geq 1$  be an integer. Then, for any choice of the nonnegative integers  $s_1, \dots, s_{q+1}$  and  $k_1, k_2$ ,*

$$\begin{aligned} \mathbb{P}(Q = q, S_1 = s_1, \dots, S_{q+1} = s_{q+1}, L'' - L' = k_1 + 1, R'' - R' = k_2 + 1 \mid \mathcal{B}_r^\bullet(\mathcal{U})) \\ = \frac{1}{4} \mathbf{1}_{\{s_1 + \dots + s_{q+1} = \mathbf{P}_r\}} \theta(s_1 + k_1) \theta(s_2) \theta(s_3) \cdots \theta(s_q) \theta(s_{q+1} + k_2). \end{aligned}$$

**Remark 5.** *Note that the above conditional distribution depends on  $\mathcal{B}_r^\bullet(\mathcal{U})$  only through its internal perimeter  $\mathbf{P}_r$ . This can be interpreted via the spatial Markov property of the UHPT.*

*Proof.* We again write  $\alpha = 12$  and  $\rho = 12\sqrt{3}$  to simplify notation. Fix a triangulation with a boundary  $A$ , with two distinguished vertices on the boundary as previously, such that  $\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A) > 0$ . Let  $p$  be the number of internal edges of  $\partial A$ . The proposition will follow if we can verify that, for any choice of the nonnegative integers  $s_1, \dots, s_{q+1}$  and  $k_1, k_2$  such that  $s_1 + \dots + s_{q+1} = p$ ,

$$\begin{aligned} \mathbb{P}(Q = q, S_1 = s_1, \dots, S_{q+1} = s_{q+1}, L'' - L' = k_1 + 1, R'' - R' = k_2 + 1 \mid \mathcal{B}_r^\bullet(\mathcal{U}) = A) \\ = \frac{1}{4} \theta(s_1 + k_1) \theta(s_2) \theta(s_3) \cdots \theta(s_q) \theta(s_{q+1} + k_2). \end{aligned}$$

The left-hand side of the preceding display can be written as

$$\sum_{A'} \mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A' \mid \mathcal{B}_r^\bullet(\mathcal{U}) = A)$$

where the sum is over all triangulations with a boundary  $A'$  having two distinct distinguished boundary vertices (also distinct from the root vertex) that satisfy the following properties:

- $A \subset A'$  in the sense explained in the proof of Lemma 10;
- $\tilde{\partial}A \subset \tilde{\partial}A'$ , and there are  $k_1 + 1$  boundary edges, resp.  $k_2 + 1$  boundary edges, between the left-most vertex of  $\tilde{\partial}A$  and the left-most vertex of  $\tilde{\partial}A'$ , resp. between the right-most vertex of  $\tilde{\partial}A$  and the right-most vertex of  $\tilde{\partial}A'$ ;
- $\partial A' \setminus \tilde{\partial}A'$  has  $q$  edges, each of which is incident to a “downward triangle” whose third vertex is incident to  $\partial A \setminus \tilde{\partial}A$ , and the configuration of these downward triangles is characterized by the numbers  $s_1, \dots, s_{q+1}$  as explained before the proposition (see also Fig. 8).

Fix  $A'$  satisfying the preceding properties, and note that

$$\mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A' \mid \mathcal{B}_r^\bullet(\mathcal{U}) = A) = \frac{\mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A')}{\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A)},$$

because the event  $\{\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A'\}$  is contained in  $\{\mathcal{B}_r^\bullet(\mathcal{U}) = A\}$ . We use Lemma 10 to evaluate the ratio of the probabilities appearing in the previous display. It readily follows that

$$\frac{\mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A')}{\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A)} = \alpha^{q-p} \alpha^{-(k_1+k_2+2)} \rho^{-N} \quad (39)$$

where  $N$  denotes the number of vertices of  $A'$  that are not vertices of  $A$  or of  $\partial\mathcal{U}$ .

At this point we note that the triangulation  $A'$  is completely determined if in addition to the preceding properties we know the triangulations with a boundary that fill in the slots of  $A' \setminus A$  left by the downward triangles. Note that there are  $q + 1$  such slots, and that the first one and the last one play a particular role since their boundary contains edges of  $\tilde{\partial}A$ . More precisely, for  $2 \leq i \leq q$ , the boundary of the slot contains  $s_i + 2$  edges, whereas for  $i = 1$  it contains  $s_1 + k_1 + 2$  edges, and for  $i = q + 1$  it contains  $s_{q+1} + k_2 + 2$  edges. Write  $\mathcal{M}_i$  for the triangulations with a boundary filling the  $i$ -th slot, and let  $\text{Inn}(\mathcal{M}_i)$  be the number of inner vertices of  $\mathcal{M}_i$ . Then, we have

$$N = q - 1 + \sum_{i=1}^{q+1} \text{Inn}(\mathcal{M}_i).$$

Set  $\tilde{s}_i = s_i$  if  $2 \leq i \leq q$  and  $\tilde{s}_1 = s_1 + k_1$ ,  $\tilde{s}_{q+1} = s_{q+1} + k_2$  to simplify notation. Then simple manipulations show that formula (39) can be rewritten in the form

$$\frac{\mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A')}{\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A)} = \alpha^{-3} \rho^2 \prod_{i=1}^{q+1} \left( \frac{1}{\rho} \alpha^{-(\tilde{s}_i-1)} \rho^{-\text{Inn}(\mathcal{M}_i)} \right)$$

(observe that  $\sum_{i=1}^{q+1} (\tilde{s}_i - 1) = p - (q + 1) + k_1 + k_2$ ). Next note that  $\alpha^{-3} \rho^2 = \frac{1}{4}$  and recall the definition (15) of the probability distribution  $\theta$ . We arrive at the formula

$$\frac{\mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A')}{\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A)} = \frac{1}{4} \prod_{i=1}^{q+1} \left( \theta(\tilde{s}_i) \frac{\rho^{-\text{Inn}(\mathcal{M}_i)}}{Z(\tilde{s}_i + 2)} \right). \quad (40)$$

It remains to sum over all possible choices of  $A'$ . But as explained earlier, this amounts to summing over possible choices of the triangulations  $\mathcal{M}_1, \dots, \mathcal{M}_{q+1}$  with boundaries of the prescribed lengths. By the very definition of  $Z(\cdot)$ , we obtain

$$\sum_{A'} \frac{\mathbb{P}(\mathcal{B}_{r+1}^\bullet(\mathcal{U}) = A')}{\mathbb{P}(\mathcal{B}_r^\bullet(\mathcal{U}) = A)} = \frac{1}{4} \prod_{i=1}^{q+1} \theta(\tilde{s}_i),$$

which completes the proof.  $\square$

**Remark 6.** A simplified version of the arguments of the preceding proof also gives the distribution of the hull  $\mathcal{B}_1^\bullet(\mathcal{U})$ . Let  $Q$  denote the number of internal edges of  $\partial\mathcal{B}_1^\bullet(\mathcal{U})$ , and write  $(R, 0)$ , respectively  $(-L, 0)$ , for the right-most vertex, resp. the left-most vertex, of  $\partial\mathcal{B}_1^\bullet(\mathcal{U}) \cap \partial\mathcal{U}$ . Then, for every  $q \geq 1$  and  $k_1, k_2 \geq 0$ ,

$$\mathbb{P}(Q = q, L = k_1 + 1, R = k_2 + 1) = \frac{1}{4} \theta(0)^{q-1} \theta(k_1) \theta(k_2) \quad (41)$$

(note that  $\theta(0) = 3/4$ ). We leave the details to the reader.

**Corollary 12.** For every integers  $c, k \geq 0$ ,

$$\mathbb{P}(S_{Q+1} = c, R'' - R' = k + 1 \mid \mathcal{B}_r^\bullet(\mathcal{U})) = \mathbf{1}_{\{c \leq \mathbf{P}_r\}} \frac{1}{2} (1 + h(\mathbf{P}_r - c)) \theta(c + k), \quad (42)$$

where we recall that  $h(j) = 4^{-j} \binom{2j}{j}$  for every integer  $j \geq 0$ . Consequently, for every integer  $k \geq 0$ ,

$$\mathbb{P}(R'' - R' = k + 1 \mid \mathcal{B}_r^\bullet(\mathcal{U})) \leq \theta([k, \infty)). \quad (43)$$

*Proof.* From the identity of Proposition 11, we get that

$$\begin{aligned} & \mathbb{P}(S_{Q+1} = c, R'' - R' = k + 1 \mid \mathcal{B}_r^\bullet(\mathcal{U})) \\ &= \frac{1}{4} \left( \sum_{q=1}^{\infty} \sum_{s_1 + \dots + s_q = \mathbf{P}_r - c} \theta([s_1, \infty)) \theta(s_2) \dots \theta(s_{q-1}) \theta(s_q) \right) \theta(c + k). \end{aligned}$$

where the right-hand side is 0 if  $\mathbf{P}_r < c$ . We first observe that, for every integer  $p \geq 0$ ,

$$\sum_{q=1}^{\infty} \sum_{s_1 + \dots + s_q = p} \theta([s_1, \infty)) \theta(s_2) \dots \theta(s_{q-1}) \theta(s_q) = 1. \quad (44)$$

This identity is immediate: If  $X_1, X_2, \dots$  is a sequence of i.i.d. random variables with distribution  $\theta$ , and if  $H_p := \min\{q \geq 1 : X_1 + \dots + X_q > p\}$ , the left-hand side of (44) is just  $\sum_{q=1}^{\infty} \mathbb{P}(H_p = q) = 1$ . To get formula (42), we then need to verify that, for every integer  $p \geq 0$ ,

$$\sum_{q=1}^{\infty} \sum_{s_1 + \dots + s_q = p} \theta(s_1) \theta(s_2) \dots \theta(s_{q-1}) \theta(s_q) = 1 + 2h(p). \quad (45)$$

For  $p = 0$ , this follows immediately from the fact that  $\theta(0) = 3/4$ . So we restrict our attention to  $p \geq 1$ . We first observe that

$$\sum_{s_1 + \dots + s_q = p} \theta(s_1) \theta(s_2) \dots \theta(s_{q-1}) \theta(s_q) = Q_q(0, p)$$

where  $Q_q(i, j)$  denotes the transition kernel of the random walk with jump distribution  $\theta$ . Hence, for  $p \geq 1$ , the left-hand side of (45) is equal to  $G(0, p)$ , where  $G(i, j) = \sum_{q=0}^{\infty} Q_q(i, j)$  is the Green kernel of the same random walk. We then observe that, for  $x \in [0, 1)$ ,

$$\sum_{k=0}^{\infty} x^k G(0, k) = \sum_{q=0}^{\infty} g_{\theta}(x)^q = \frac{1}{1 - g_{\theta}(x)} = \left(1 + \frac{1}{\sqrt{1-x}}\right)^2.$$

By expanding  $(1 + \frac{1}{\sqrt{1-x}})^2$  as a power series, we obtain that, for every  $k \geq 0$ ,

$$G(0, k) = \mathbf{1}_{\{k=0\}} + 1 + 2h(k),$$

giving the right-hand side of (45) when  $k = p \geq 1$ . This completes the proof of (42). The bound (43) follows by summing over  $c$ , noting that  $h(j) \leq 1$  for every  $j \geq 0$ .  $\square$



## 4.2 Distances along the boundary of the UHPT

We now use the results of the preceding subsection to get bounds on distances between vertices of  $\partial\mathcal{U}$  and the root vertex. Recall that  $\partial\mathcal{U}$  is identified to  $\mathbb{Z} \times \{0\}$ , so that  $(0, 0)$  is the root vertex.

For every integer  $r \geq 1$ , we now write  $(-L_r^\mathcal{U}, 0)$  for the left-most vertex in  $\partial\mathcal{B}_r^\bullet(\mathcal{U}) \cap \partial\mathcal{U}$  and  $(0, R_r^\mathcal{U})$  for the right-most vertex in  $\partial\mathcal{B}_r^\bullet(\mathcal{U}) \cap \partial\mathcal{U}$ . We also set  $R_0^\mathcal{U} = 0$  and  $L_0^\mathcal{U} = 0$  by convention, and we also agree that  $\mathcal{B}_0^\bullet(\mathcal{U})$  is the edge-triangulation. Recall that  $d_{\text{gr}}^\mathcal{U}$  is the graph distance on the vertex set of  $\mathcal{U}$ .

**Proposition 13.** *The sequences  $(r^{-2}L_r^\mathcal{U})_{r \geq 1}$  and  $(r^{-2}R_r^\mathcal{U})_{r \geq 1}$  are bounded in probability: For every  $\varepsilon > 0$ , there exists a constant  $K$  such that*

$$\sup_{r \geq 1} \mathbb{P}(L_r^\mathcal{U} \geq K r^2) \leq \varepsilon \quad \text{and} \quad \sup_{r \geq 1} \mathbb{P}(R_r^\mathcal{U} \geq K r^2) \leq \varepsilon,$$

and consequently, for every  $r \geq 1$ ,

$$\mathbb{P}\left(\min_{|j| \geq K r^2} d_{\text{gr}}^\mathcal{U}((0, 0), (j, 0)) > r\right) \geq 1 - 2\varepsilon. \quad (46)$$

For every integer  $m \geq 1$ , set  $T_m := \min\{r \geq 1 : R_r^\mathcal{U} > m\}$ . There exists a constant  $K'$  such that, for every  $m \geq 1$  and every  $j \geq 1$ ,

$$\mathbb{P}(R_{T_m}^\mathcal{U} - m > j) \leq K' \sqrt{\frac{m}{m+j}}.$$

*Proof.* Let us start with the first statement. By symmetry, it suffices to consider the sequence  $(r^{-2}R_r^\mathcal{U})_{r \geq 1}$ . By combining (43) and (11), we get the existence of a constant  $H$  such that, for every  $r \geq 1$ , and every  $k \geq 1$ ,

$$\mathbb{P}(R_{r+1}^\mathcal{U} - R_r^\mathcal{U} = k \mid R_1^\mathcal{U}, \dots, R_r^\mathcal{U}) \leq H k^{-3/2}. \quad (47)$$

From this bound, for instance by using a coupling with a stable subordinator with index  $1/2$ , one derives the fact that the sequence  $(r^{-2}R_r^\mathcal{U})_{r \geq 1}$  is bounded in probability. We then note that the condition  $j > R_r^\mathcal{U}$ , or  $j < -L_r^\mathcal{U}$  implies by definition that  $(j, 0) \notin \mathcal{B}_r^\bullet(\mathcal{U})$ . Therefore,

$$\min_{|j| > R_r^\mathcal{U} \vee L_r^\mathcal{U}} d_{\text{gr}}^\mathcal{U}((0, 0), (j, 0)) > r,$$

giving (46) since, by the first part of the proposition,

$$\mathbb{P}(R_r^\mathcal{U} \vee L_r^\mathcal{U} < K r^2) \geq 1 - 2\varepsilon.$$

Let us turn to the proof of the last assertion. It is enough to establish the existence of a constant  $K'$ , which does not depend on  $m$ , such that, for every integer  $k \geq 0$ , for every  $\ell \in \{0, 1, \dots, m\}$  and for every integer  $j \geq 0$ ,

$$\mathbb{P}(R_{k+1}^\mathcal{U} - R_k^\mathcal{U} > \ell + j \mid \mathcal{B}_k^\bullet(\mathcal{U})) \leq K' \sqrt{\frac{m}{m+j}} \mathbb{P}(R_{k+1}^\mathcal{U} - R_k^\mathcal{U} > \ell \mid \mathcal{B}_k^\bullet(\mathcal{U})). \quad (48)$$

Indeed, if (48) holds, we have, for every  $j \geq 0$ ,

$$\begin{aligned}
\mathbb{P}(R_{T_m}^{\mathcal{U}} - m > j) &= \sum_{k=0}^{\infty} \mathbb{P}(R_k^{\mathcal{U}} \leq m, R_{k+1}^{\mathcal{U}} - R_k^{\mathcal{U}} > m + j - R_k^{\mathcal{U}}) \\
&= \sum_{k=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{R_k^{\mathcal{U}} \leq m\}} \mathbb{P}(R_{k+1}^{\mathcal{U}} - R_k^{\mathcal{U}} > m + j - R_k^{\mathcal{U}} \mid \mathcal{B}_k^{\bullet}(\mathcal{U}))] \\
&\leq K' \sqrt{\frac{m}{m+j}} \sum_{k=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{R_k^{\mathcal{U}} \leq m\}} \mathbb{P}(R_{k+1}^{\mathcal{U}} - R_k^{\mathcal{U}} > m - R_k^{\mathcal{U}} \mid \mathcal{B}_k^{\bullet}(\mathcal{U}))] \\
&= K' \sqrt{\frac{m}{m+j}}.
\end{aligned}$$

Let us prove (48). The case  $k = 0$  follows very easily from (41) and so we concentrate on the case  $k \geq 1$ . From Corollary 12, we have, for every  $j \geq 0$ ,

$$\mathbb{P}(R_{k+1}^{\mathcal{U}} - R_k^{\mathcal{U}} > \ell + j \mid \mathcal{B}_k^{\bullet}(\mathcal{U})) = \frac{1}{2} \sum_{i=\ell+j}^{\infty} \sum_{c=0}^{\mathbf{P}_k} (1 + h(\mathbf{P}_k - c)) \theta(c+i) = \frac{1}{2} \sum_{c=0}^{\mathbf{P}_k} (1 + h(\mathbf{P}_k - c)) \theta([c + \ell + j, \infty)),$$

where  $\mathbf{P}_k$  is the number of internal edges of  $\partial \mathcal{B}_k^{\bullet}(\mathcal{U})$ . Since  $1 \leq 1 + h(\mathbf{P}_k - c) \leq 2$ , the bound (48) will follow if we can verify that, for every  $p \geq 1$ , for every  $\ell \in \{0, 1, \dots, m\}$  and every  $j \geq 0$ ,

$$\sum_{c=0}^p \theta([c + \ell + j, \infty)) \leq K'' \sqrt{\frac{m}{m+j}} \sum_{c=0}^p \theta([c + \ell, \infty)),$$

with a suitable constant  $K''$ . The bound in the last display is derived by elementary arguments relying on (11): Note that, if  $0 \leq c \leq m$ , the ratio  $\theta([c + \ell + j, \infty))/\theta([c + \ell, \infty))$  is bounded above by (a constant times)  $(m/(m+j))^{3/2}$ , and on the other hand

$$\sum_{c=m}^{\infty} \theta([c + \ell + j, \infty))$$

is bounded above by a constant times  $(m+j)^{-1/2}$ , whereas

$$\sum_{c=0}^m \theta([c + \ell, \infty))$$

is bounded below by a positive constant times  $m^{-1/2}$ .  $\square$

We will need a reinforcement of property (46), which is given in the next proposition.

**Proposition 14.** *Let  $\varepsilon > 0$ . For every integer  $A > 0$ , we can choose an integer  $K > 0$  sufficiently large so that, for every  $r \geq 1$ ,*

$$\mathbb{P}\left(\min_{0 \leq i \leq Ar^2, j \geq Kr^2} d_{\text{gr}}^{\mathcal{U}}((i, 0), (j, 0)) \leq r\right) \leq \varepsilon.$$

*Proof.* Fix an integer  $A > 0$ . By the last assertion of the previous proposition, we can choose an integer  $A' > A$  large enough so that, for every  $r \geq 1$ ,

$$\mathbb{P}\left(R_{T_{Ar^2}}^{\mathcal{U}} \geq A'r^2\right) < \frac{\varepsilon}{2}.$$

Fix  $r \geq 1$  and set, for every  $j \geq 0$ ,

$$\tilde{R}_j^{\mathcal{U}} := R_{T_{Ar^2}+j}^{\mathcal{U}} - R_{T_{Ar^2}}^{\mathcal{U}}.$$

Since  $T_{Ar^2}$  is a stopping time of the process  $(R_j^\mathcal{U})_{j \geq 0}$ , it follows from (47) that we have also, for every  $j \geq 0$  and  $k \geq 1$ ,

$$\mathbb{P}(\tilde{R}_{j+1}^\mathcal{U} - \tilde{R}_j^\mathcal{U} = k \mid \tilde{R}_1^\mathcal{U}, \dots, \tilde{R}_j^\mathcal{U}) \leq H k^{-3/2}.$$

As in the proof of Proposition 13, this implies that the sequence  $(j^{-2} \tilde{R}_j^\mathcal{U})_{j \geq 1}$  is bounded in probability, and since  $H$  does not depend on our choice of  $r$ , we even get bounds that are uniform in  $r$ . Hence we can choose an integer  $K > A'$ , which does not depend on the choice of  $r$ , so that

$$\mathbb{P}(\tilde{R}_r^\mathcal{U} \geq (K - A')r^2) < \frac{\varepsilon}{2}.$$

Finally, we know that any vertex  $(i, 0)$  with  $0 \leq i \leq Ar^2$  belongs to the hull of radius  $T_{Ar^2}$  (because by definition  $R_{T_{Ar^2}}^\mathcal{U} > Ar^2$ ). On the event  $\{R_{T_{Ar^2}}^\mathcal{U} < A'r^2\} \cap \{\tilde{R}_r^\mathcal{U} < (K - A')r^2\}$ , we have  $R_{T_{Ar^2}+r}^\mathcal{U} < Kr^2$ , which implies that vertices  $(j, 0)$  with  $j \geq Kr^2$  do not belong to the hull of radius  $T_{Ar^2} + r$ . Hence, on the latter event we must have  $d_{\text{gr}}^\mathcal{U}((i, 0), (j, 0)) > r$  whenever  $0 \leq i \leq Ar^2$  and  $j \geq Kr^2$ . This completes the proof.  $\square$

**Remark 7.** The known results for quadrangulations [14] strongly suggest that  $(r^{-2}L_{[tr]}^\mathcal{U}, r^{-2}R_{[tr]}^\mathcal{U})_{t \geq 0}$  should converge in distribution to the pair formed by the last hitting time processes of two independent Bessel processes of dimension 5.

### 4.3 Distances along the boundary of the LHPT

We will now deduce an analog of Proposition 14 for distances along the boundary of the LHPT  $\mathcal{L}$ . Our main technical tool will be the absolute continuity property of Proposition 8. We will also use left-most geodesics, which are defined in a way similar to the end of Section 2.2. For every  $i \in \mathbb{Z}$  and every integer  $r \geq 1$ , the left-most geodesic from  $(i, 0)$  in  $\mathcal{L}$  is the infinite geodesic path  $\omega$  in  $\mathcal{L}$  that starts from  $\omega(0) = (i, 0)$ , visits a vertex  $\omega(n) \in \mathcal{L}_n$  at each step  $n \geq 0$ , and is obtained by choosing at step  $n + 1$  the left-most edge between  $\omega(n)$  and  $\mathcal{L}_{n+1}$ . For every  $r \geq 1$ , the first  $r$  edges on this path give the left-most geodesic from  $(i, 0)$  to  $\mathcal{L}_r$  in  $\mathcal{L}$ . Similarly, in the UHPT  $\mathcal{U}$ , we can define the left-most geodesic from  $(i, r)$  to  $\partial\mathcal{U}$ , for every  $i \in \mathbb{Z}$  and  $r \geq 1$ . Furthermore, if  $1 \leq i < j$ , the left-most geodesics from  $(i, r)$  to  $\partial\mathcal{U}$  and from  $(j, r)$  to  $\partial\mathcal{U}$  coalesce before hitting  $\partial\mathcal{U}$  (possibly when they hit  $\partial\mathcal{U}$ ), if and only if none of the trees  $\Gamma_{(i,r)}, \Gamma_{(i+1,r)}, \dots, \Gamma_{(j-1,r)}$  has height  $r$ .

Recall that  $d_{\text{gr}}^\mathcal{L}$  is the graph distance on the vertex set of  $\mathcal{L}$ .

**Proposition 15.** For every  $\varepsilon > 0$ , there exists an integer  $K \geq 1$  such that for every  $r \geq 1$ ,

$$\mathbb{P}\left(\min_{|j| \geq Kr^2} d_{\text{gr}}^\mathcal{L}((0, 0), (j, 0)) \geq r\right) \geq 1 - \varepsilon.$$

Consequently, we have also with  $K' = 4K$ , for every  $r \geq 1$ ,

$$\mathbb{P}\left(\min_{|j| \geq 2K'r^2} \min_{-K'r^2 \leq i \leq K'r^2} d_{\text{gr}}^\mathcal{L}((i, 0), (j, 0)) \geq r\right) \geq 1 - 2\varepsilon.$$

*Proof.* Let us start with the first assertion. For obvious symmetry reasons, it is enough to consider positive values of  $j$ . For every integer  $K \geq 1$ , and for every  $r \geq 1$ , we consider the measurable set  $A_r(K)$  such that  $\mathcal{L}_{[0,r]} \in A_r(K)$  if and only if there is a path in  $\mathcal{L}_{[0,r]}$  of length strictly smaller than  $r$  that connects  $(0, 0)$  to  $(0, j)$ , for some  $j \geq Kr^2$ . Note that  $\mathcal{L}_{[0,r]} \in A_r(K)$  if and only if

$$\min_{j \geq Kr^2} d_{\text{gr}}^\mathcal{L}((0, 0), (j, 0)) < r,$$

since a path in  $\mathcal{L}$  that starts from  $(0,0)$  and has length strictly smaller than  $r$  must stay in  $\mathcal{L}_{[0,r]}$ . We therefore need to prove that, if  $K$  is chosen sufficiently large, we have, for every  $r \geq 1$ ,

$$\mathbb{P}(\mathcal{L}_{[0,r]} \in A_r(K)) < \varepsilon.$$

Thanks to Corollary 9, it is then enough to verify that, if  $K$  is sufficiently large, we have for every  $r \geq 1$ ,

$$\mathbb{P}(\tilde{\mathcal{U}}_{[0,r]} \in A_r(K)) < \varepsilon,$$

where  $\tilde{\mathcal{U}}_{[0,r]}$  is defined in Proposition 8. Recalling the notation of this proposition, we have

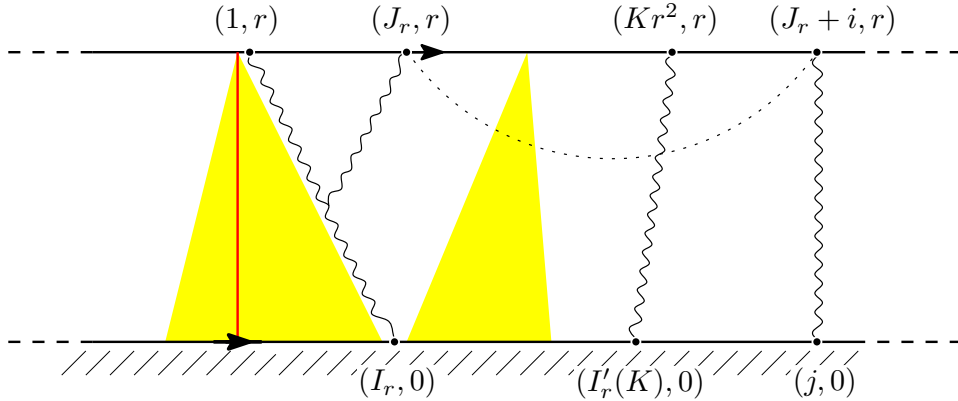
$$\{\tilde{\mathcal{U}}_{[0,r]} \in A_r(K)\} \subset \left\{ \min_{i \geq Kr^2} d_{\text{gr}}^{\mathcal{U}}((J_r, r), (J_r + i, r)) < r \right\}.$$

To bound the probability of the event in the right-hand side, let  $(I_r, 0)$  be the endpoint of the left-most geodesic from  $(1, r)$  to  $\partial\mathcal{U}$  in  $\mathcal{U}$ . By the observations preceding the proposition, and the fact that  $1 \leq J_r \leq Kr$ , we get that the left-most geodesic from  $(J_r, r)$  to  $\partial\mathcal{U}$  coalesces with the one from  $(1, r)$  before reaching  $\partial\mathcal{U}$  (possibly when hitting  $\partial\mathcal{U}$ ). Consequently, we have

$$d_{\text{gr}}^{\mathcal{U}}((J_r, r), (I_r, 0)) = r.$$

Let  $(I'_r(K), 0)$  be the endpoint of the left-most geodesic from  $(Kr^2, r)$  to  $\partial\mathcal{U}$  in  $\mathcal{U}$ . See Fig. 9 for an illustration of the preceding definitions. Suppose that both  $J_r < Kr^2$  and there exists  $i \geq Kr^2$  such that  $d_{\text{gr}}^{\mathcal{U}}((J_r, r), (J_r + i, r)) < r$ . Then the endpoint of the left-most geodesic from  $(J_r + i, r)$  to  $\partial\mathcal{U}$  is of the form  $(j, 0)$  with  $j \geq I'_r(K)$  since  $J_r + i \geq Kr^2$ . The triangle inequality then shows that

$$d_{\text{gr}}^{\mathcal{U}}((I_r, 0), (j, 0)) \leq d_{\text{gr}}^{\mathcal{U}}((I_r, 0), (J_r, r)) + d_{\text{gr}}^{\mathcal{U}}((J_r, r), (J_r + i, r)) + d_{\text{gr}}^{\mathcal{U}}((J_r + i, r), (j, 0)) < 3r.$$



**Figure 9:** Illustration of the Proof of Proposition 15. The wavy lines represent left-most geodesics, the dotted line is the unlikely path linking  $(J_r, r)$  to  $(J_r + i, r)$ . Finally the first two trees  $\Gamma_{(\ell, r)}$  with  $\ell \geq 0$  that hit  $\partial\mathcal{U}$  are represented in yellow.

Summarizing, we have

$$\mathbb{P}(\tilde{\mathcal{U}}_{[0,r]} \in A_r(K)) \leq \mathbb{P}(J_r \geq Kr^2) + \mathbb{P}\left(\min_{j \geq I'_r(K)} d_{\text{gr}}^{\mathcal{U}}((I_r, 0), (j, 0)) < 3r\right). \quad (49)$$

Since we already noticed that  $\mathbb{E}[(J_r)^2] \leq 4(r+1)^4$ , the Markov inequality immediately tells us that we can find  $K_0$  such that  $\mathbb{P}(J_r \geq Kr^2) \leq \varepsilon/4$  if  $K \geq K_0$ . We then bound the second term in the right-hand

side of (49). We first claim that, we can find an integer  $B$  such that, for every  $r \geq 1$ ,

$$\mathbb{P}(I_r \geq Br^2) \leq \frac{\varepsilon}{4}. \quad (50)$$

Indeed, from the construction of the downward triangles of the UHPT in Section 3.1, it is not hard to see that  $I_r$  is equal to the maximal integer  $m \geq 1$  such that  $(m - \frac{1}{2}, 0)$  is a vertex of  $\Gamma_{(0,r)}$ . Consequently,  $I_r$  is bounded above by the size  $N_r$  of generation  $r$  of the tree  $\Gamma_{(0,r)}$ , which is a size-biased Galton–Watson tree with offspring distribution  $\theta$ . Note that, for  $x \in [0, 1)$ ,

$$\mathbb{E}[x^{N_r}] = \sum_{k=1}^{\infty} k \mathbb{P}_1(Y_r = k) x^k = x \times \frac{d}{dx} \left( 1 - \left( r + \frac{1}{\sqrt{1-x}} \right)^{-2} \right) = x(1 + r\sqrt{1-x})^{-3}$$

from which one easily gets that  $r^{-2}N_r$  converges in distribution, yielding the estimate (50).

We then use Proposition 14, which allows us to find an integer  $K$  sufficiently large so that, for every  $r \geq 1$ ,

$$\mathbb{P} \left( \min_{0 \leq i \leq Br^2, j \geq Kr^2/3} d_{\text{gr}}^{\mathcal{U}}((i, 0), (j, 0)) \leq 3r \right) \leq \frac{\varepsilon}{4}. \quad (51)$$

We finally claim that, by choosing  $K$  even larger if needed, we have also for every  $r \geq 1$ ,

$$\mathbb{P}(I'_r(K) < Kr^2/3) \leq \frac{\varepsilon}{4}. \quad (52)$$

Assuming that (52) holds, we can combine (50), (51) and (52) to get that the second term in the right-hand side of (49) is bounded above by  $3\varepsilon/4$ , and since we also know that  $\mathbb{P}(J_r \geq Kr^2) \leq \varepsilon/4$ , we deduce from (49) that  $\mathbb{P}(\tilde{\mathcal{U}}_{[0,r]} \in A_r(K)) \leq \varepsilon$ , which was the desired result for the first assertion of the proposition.

It remains to verify that the estimate (52) holds. From the construction of the UHPT, the quantity  $I'_r(K)$  is bounded below by

$$M_r(K) := \sum_{\ell=1}^{Kr^2-1} \#\Gamma_{(\ell,r)}(r).$$

We have, for every  $x \in [0, 1)$ ,

$$\mathbb{E}[x^{M_r(K)}] = \left( 1 - \left( r + \frac{1}{\sqrt{1-x}} \right)^{-2} \right)^{Kr^2-1},$$

and it follows by straightforward calculations that  $r^{-2}M_r(K)$  converges in distribution to a random variable  $U_K$  with Laplace transform  $\mathbb{E}[e^{-\lambda U_K}] = \exp(-K(1 + \lambda^{-1/2})^{-2})$ . Then  $U_K/K$  converges in probability to 1 as  $K \rightarrow \infty$ , and thus we can fix  $K$  sufficiently large so that  $\mathbb{P}(U_K > K/2) > 1 - \varepsilon/4$ . For this value of  $K$ , the estimate (52) holds for all sufficiently large  $r$ . We can deal with the remaining values of  $r$  by taking  $K$  even larger if necessary (using now the law of large numbers). This completes the proof of the first assertion.

The proof of the second assertion is now easy. If we assume that for some  $j \geq 2K'r^2$  and for some  $i \in \{-K'r^2, \dots, K'r^2\}$ , we have  $d_{\text{gr}}^{\mathcal{L}}((i, 0), (j, 0)) < r$ , then any geodesic from  $(i, 0)$  to  $(j, 0)$  must stay in  $\mathcal{L}_{[0,r]}$  (otherwise it has length greater than  $r$ ) and by a planarity argument it must intersect the leftmost geodesic from  $(K'r^2, 0)$  to the horizontal line  $\mathcal{L}_r$ , and it follows that  $d_{\text{gr}}^{\mathcal{L}}((K'r^2, 0), (j, 0)) < 2r$ . However, by the first assertion of the proposition (with  $r$  replaced by  $2r$ ) and a translation argument, the probability that  $d_{\text{gr}}^{\mathcal{L}}((K'r^2, 0), (j, 0)) < 2r$  for some  $j \geq 2K'r^2$  is bounded above by  $\varepsilon$ . The desired result follows, noting that we must also consider the case  $j < -2K'r^2$ .  $\square$

The preceding proposition provides lower bounds on distances between vertices on the boundary of the LHPT. We state another proposition that gives upper bounds for the same quantities. We recall the notation  $\mathcal{L}_i$  for the line  $\{-i\} \times \mathbb{Z}$ . Also recall the definition of the left-most geodesic in  $\mathcal{L}$  starting from  $v$ , for every  $v \in \partial\mathcal{L}$ .

**Proposition 16.** *Let  $\delta > 0$  and  $\gamma > 0$ . We can choose an integer  $A \geq 1$  such that the following holds for every sufficiently large  $n$ . With probability at least  $1 - \delta$ , we have:*

- *for every  $i \in \{-n+1, -n+2, \dots, n\}$ , the left-most geodesic starting from  $(i, 0)$  coalesces with the left-most geodesic starting from  $(-n + \lfloor 2\ell n/A \rfloor, 0)$ , for some  $0 \leq \ell \leq A$ , before hitting  $\mathcal{L}_{\lfloor \gamma\sqrt{n} \rfloor}$ ;*
- *for every  $i, j \in \{-n+1, -n+2, \dots, n\}$ , with  $i < j$ , there is a path from  $(i, 0)$  to  $(j, 0)$  that stays in  $\mathcal{L}_{[0, \lfloor \gamma\sqrt{n} \rfloor]}$  and has length smaller than*

$$\left( \left\lfloor \frac{A(j-i)}{2n} \right\rfloor + 2 \right) (1 + 2\gamma\sqrt{n}).$$

*Proof.* Let  $U_1^{(n)} < U_2^{(n)} < \dots < U_{m_n}^{(n)}$  be all indices in  $\{1, 2, \dots, 2n-1\}$  such that the height of  $\mathcal{T}_{-n+i}$  is greater than or equal to  $\lfloor \gamma\sqrt{n} \rfloor$ . If we set for every  $t \in [0, 2n]$ ,

$$N_t^{(n)} := \#\{i \in \{1, \dots, m_n\} : U_i^{(n)} \leq t\},$$

it follows from (28) that  $(N_{\lfloor nt \rfloor}^{(n)})_{0 \leq t \leq 2}$  converges in distribution in the Skorokhod sense to a Poisson process with parameter  $\gamma^{-2}$ . Write  $U_0^{(n)} = 0$  and  $U_{m_n+1}^{(n)} = 2n$  by convention. The preceding observations imply that we can choose  $\eta > 0$  small enough so that, for every sufficiently large  $n$ , the property

$$U_{i+1}^{(n)} - U_i^{(n)} > \eta n, \quad \forall i \in \{0, 1, \dots, m_n\} \quad (53)$$

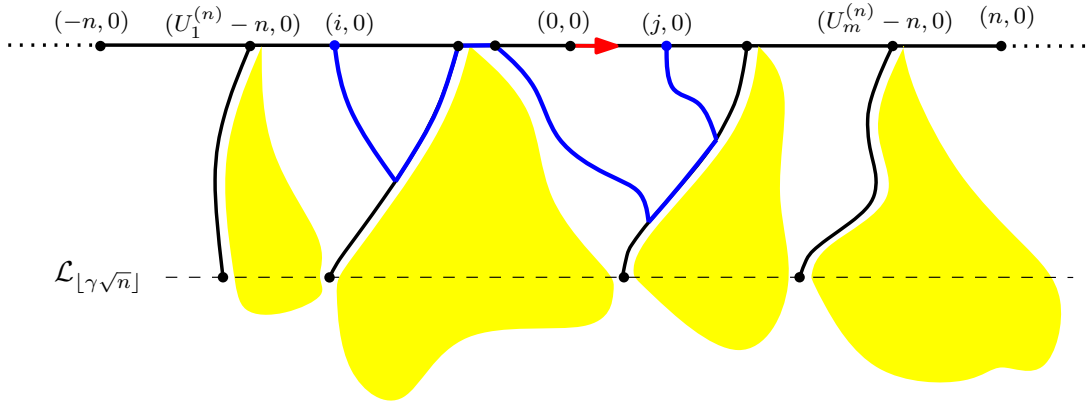
holds with probability at least  $1 - \delta$ .

By the coalescence property of left-most geodesics, if  $U_j^{(n)} < i \leq i' \leq U_{j+1}^{(n)}$ , the leftmost geodesic from  $(-n+i, 0)$  coalesces with that from  $(-n+i', 0)$  before hitting the line  $\mathcal{L}_{\lfloor \gamma\sqrt{n} \rfloor}$ . Now set  $A = \lfloor 2/\eta \rfloor + 1$ , so that  $2/A < \eta$ . On the event where (53) holds, each interval  $]U_j^{(n)}, U_{j+1}^{(n)}]$ , for  $0 < j \leq m_n$ , contains at least one of the points  $\lfloor 2\ell n/A \rfloor$ ,  $1 \leq \ell \leq A$ . The first assertion of the proposition follows.

To get the second assertion, let  $k \leq \ell$  be such that  $U_k^{(n)} + 1 \leq i \leq U_{k+1}^{(n)}$  and  $U_\ell^{(n)} + 1 \leq j \leq U_{\ell+1}^{(n)}$ . If  $k = \ell$ , the left-most geodesics from  $(i, 0)$  and  $(j, 0)$  coalesce before hitting  $\mathcal{L}_{\lfloor \gamma\sqrt{n} \rfloor}$ , and we construct a path from  $(i, 0)$  to  $(j, 0)$ , with length bounded above by  $2\gamma\sqrt{n}$ , by concatenating the parts of these two geodesics before their coalescence time. If  $k < \ell$ , we construct a path from  $(i, 0)$  to  $(j, 0)$  as follows. We first construct a path from  $(i, 0)$  to  $(U_{k+1}^{(n)}, 0)$  with length smaller than  $2\gamma\sqrt{n}$  by concatenating left-most geodesics, and we add to this path the edge between  $(U_{k+1}^{(n)}, 0)$  and  $(U_{k+1}^{(n)} + 1, 0)$ . We then concatenate the left-most geodesics from  $(U_{k+1}^{(n)} + 1, 0)$  and from  $(U_{k+2}^{(n)}, 0)$  up to their coalescence time to get a path from  $(U_{k+1}^{(n)} + 1, 0)$  to  $(U_{k+2}^{(n)}, 0)$  with length smaller than  $2\gamma\sqrt{n}$ , and we add to this path the edge between  $(U_{k+2}^{(n)}, 0)$  and  $(U_{k+2}^{(n)} + 1, 0)$ . We continue inductively and, when we reach  $(U_\ell^{(n)} + 1, 0)$ , we add the path obtained by the concatenation of the two left-most geodesics from  $(U_\ell^{(n)} + 1, 0)$  and from  $(j, 0)$  (see Fig. 10). This construction yields a path from  $(i, 0)$  to  $(j, 0)$  with length smaller than  $(\ell - k + 1)(1 + 2\gamma\sqrt{n})$ . Finally, on the event where (53) holds, we have  $(\ell - k - 1)\eta n \leq j - i$ , hence

$$\ell - k \leq \left\lfloor \frac{j-i}{\eta n} \right\rfloor + 1 \leq \left\lfloor \frac{A(j-i)}{2n} \right\rfloor + 1,$$

giving the desired result.  $\square$



**Figure 10:** Illustration of the proof of Proposition 16. The trees reaching height  $\lfloor \gamma \sqrt{n} \rfloor$  are represented in yellow. The left-most geodesics are represented in black and the (non-geodesic) path connecting  $(i, 0)$  to  $(j, 0)$ , which is constructed in the proof, is represented in blue.

We will now derive a result similar to the preceding proposition for the UIPT  $\mathcal{T}_\infty^{(1)}$  of the 1-gon. Recall that we denote the length of  $\partial^* B_r^\bullet(\mathcal{T}_\infty^{(1)})$  by  $L_r$ .

For every integer  $n \geq 1$ , we write  $u_0^{(n)}$  for a vertex chosen uniformly at random on  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$ , and  $u_1^{(n)}, \dots, u_{L_n-1}^{(n)}$  for the other vertices of  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$  enumerated in clockwise order, starting from  $u_0^{(n)}$ . We extend the definition of  $u_i^{(n)}$  by periodicity, requiring that  $u_i^{(n)} = u_{L_n+i}^{(n)}$  for every  $i \in \mathbb{Z}$ . Note that, for every  $i \in \mathbb{Z}$ ,  $u_i^{(n)}$  is also uniformly distributed over  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$ .

**Proposition 17.** *Let  $\gamma \in (0, 1/2)$  and  $\delta > 0$ . For every integer  $A \geq 1$ , let  $H_{n,A}$  be the event where any left-most geodesic to the root starting from a vertex of  $\partial^* B_n^\bullet(\mathcal{T}_\infty^{(1)})$  coalesces before time  $\lfloor \gamma n \rfloor$  with one of the left-most geodesics to the root starting from  $u_{\lfloor kn^2/A \rfloor}^{(n)}$ ,  $0 \leq k \leq \lfloor n^{-2} L_n A \rfloor$ . Then, we can choose  $A$  large enough so that, for every sufficiently large  $n$ ,*

$$\mathbb{P}(H_{n,A}) \geq 1 - \delta.$$

*Proof.* Recall the notation introduced before Proposition 5. We may assume that the first tree in the forest  $\tilde{\mathcal{F}}_{n-\lfloor \gamma n \rfloor, n}^{(1)}$  is the tree rooted at the edge between  $u_0^{(n)}$  and  $u_1^{(n)}$ . Then write  $\tilde{\mathcal{F}}_{n-\lfloor \gamma n \rfloor, n}^{(1)} = (\tau_1^{(n)}, \dots, \tau_{L_n}^{(n)})$ . By the remarks of the end of subsection 2.2, we know that, for  $1 \leq i < j \leq n$ , the left-most geodesics to the root from  $u_i^{(n)}$  and from  $u_j^{(n)}$  coalesce before time  $\lfloor \gamma n \rfloor$  (possibly exactly at time  $\lfloor \gamma n \rfloor$ ) as soon as all the trees  $\tau_{i+1}^{(n)}, \tau_{i+2}^{(n)}, \dots, \tau_j^{(n)}$  have height strictly smaller than  $\lfloor \gamma n \rfloor$ . To verify that  $H_{n,A}$  holds, it is therefore sufficient to verify that, for any  $i \in \{1, \dots, L_n\}$  there exists an index  $k$  with  $0 \leq k \leq \lfloor n^{-2} L_n A \rfloor$  such that all trees  $\tau_j^{(n)}$  with  $\lfloor kn^2/A \rfloor \wedge i < j \leq \lfloor kn^2/A \rfloor \vee i$  have height strictly smaller than  $\lfloor \gamma n \rfloor$ . Write  $H'_{n,A}$  for the event where the latter property holds. By (23) and (22) we can find  $a > 0$  such that the event

$$\{\lfloor an^2 \rfloor < L_n \leq \lfloor a^{-1} n^2 \rfloor\} \cap \{\lfloor an^2 \rfloor < L_{n-\lfloor \gamma n \rfloor} \leq \lfloor a^{-1} n^2 \rfloor\}$$

has probability at least  $1 - \delta/2$ . On the other hand, if we want to bound the probability of the intersection of the latter event with the complement of  $H'_{n,A}$ , Proposition 5 shows that we may replace the forest  $\tilde{\mathcal{F}}_{n-\lfloor \gamma n \rfloor, n}^{(1)}$  by a forest of independent Galton–Watson trees with offspring distribution  $\theta$  truncated at height  $\lfloor \gamma n \rfloor$  (at the cost of the multiplicative constant  $C_1$ ). But then the desired result follows by the same arguments as in the first part of the proof of Proposition 16.  $\square$



## 5 First-passage percolation on the UIPT

The geometric estimates gathered in the last sections will now be used to study the behavior of modified distances in random triangulations. As explained in the Introduction, we first concentrate on the case of the first-passage percolation  $d_{\text{fpp}}$ . After establishing an easy subadditive result (Proposition 18), we derive the key result of this section (Proposition 20), which deals with the modified distance between the root vertex and an arbitrary vertex of the boundary of the hull of radius  $r$  in the UIPT.

### 5.1 Subadditivity in the lower half-plane model

We consider the LHPT  $\mathcal{L}$ , and, conditionally on  $\mathcal{L}$ , we assign i.i.d. random weights to the edges of  $\mathcal{L}$ . We assume that the common distribution of these random variables is supported on the interval  $[\kappa, 1]$  for some  $\kappa \in (0, 1]$ . From these random weights, we define the first-passage percolation distance  $d_{\text{fpp}}^{\mathcal{L}}$  as explained in the Introduction.

Recall our notation  $\mathcal{L}_r = \{(i, -r) : i \in \mathbb{Z}\}$  for the lower boundary of  $\mathcal{L}_{[0, r]}$ . Also recall that  $\rho = (0, 0)$  is the root vertex.

**Proposition 18.** *There exists a constant  $\mathbf{c}_0 \in [\kappa, 1]$  such that*

$$r^{-1} d_{\text{fpp}}^{\mathcal{L}}(\rho, \mathcal{L}_r) \xrightarrow[r \rightarrow \infty]{\text{a.s.}} \mathbf{c}_0.$$

*Proof.* For integers  $0 \leq m < n$ , we define  $\mathcal{L}_{[m, n]}$  as the part of  $\mathcal{L}$  lying in the strip  $\mathbb{R} \times [-n, -m]$ . The first-passage percolation distance  $d_{\text{fpp}}^{\mathcal{L}_{[m, n]}}$  on the vertex set of  $\mathcal{L}_{[m, n]}$  is defined by considering the minimal weight of paths that stay in  $\mathcal{L}_{[m, n]}$  (thus, if  $v$  and  $v'$  are two vertices of  $\mathcal{L}_{[m, n]}$  we have  $d_{\text{fpp}}^{\mathcal{L}}(v, v') \leq d_{\text{fpp}}^{\mathcal{L}_{[m, n]}}(v, v')$ ).

Then let  $n, m \geq 1$  and let  $x_m$  be the left-most vertex of  $\mathcal{L}_m$  such that  $d_{\text{fpp}}^{\mathcal{L}}(\rho, \mathcal{L}_m) = d_{\text{fpp}}^{\mathcal{L}}(\rho, x_m)$ . We have

$$d_{\text{fpp}}^{\mathcal{L}}(\rho, \mathcal{L}_{m+n}) \leq d_{\text{fpp}}^{\mathcal{L}}(\rho, \mathcal{L}_m) + d_{\text{fpp}}^{\mathcal{L}_{[m, m+n]}}(x_m, \mathcal{L}_{m+n}).$$

Remark that  $x_m$  is a function of  $\mathcal{L}_{[0, m]}$  only and that, by the independence of the layers in  $\mathcal{L}$ , the random variable  $d_{\text{fpp}}^{\mathcal{L}_{[m, m+n]}}(x_m, \mathcal{L}_{m+n})$  is independent of  $\mathcal{L}_{[0, m]}$  and has the same distribution as  $d_{\text{fpp}}^{\mathcal{L}}(\rho, \mathcal{L}_n)$ . We can then apply Liggett's version of Kingman's subadditive ergodic theorem [28] to get the statement of the proposition (the fact that the limit is constant is easy from a zero-one law argument, and the property  $\kappa \leq \mathbf{c}_0 \leq 1$  is obvious).  $\square$

### 5.2 From the lower half-plane to the UIPT

We now discuss the first-passage percolation distance  $d_{\text{fpp}}$  on the UIPT of the 1-gon  $\mathcal{T}_{\infty}^{(1)}$ . We assume that this distance is defined in terms of i.i.d. weights on the edges of the UIPT, these weights having the same distribution as in the previous section. Note in particular that  $d_{\text{fpp}} \leq d_{\text{gr}}$  since we assume that weights are bounded above by 1. We still write  $\mathbf{c}_0$  for the constant arising in Proposition 18.

Let us state our main technical result, which builds upon Proposition 18 and the geometric estimates of the last section. To simplify notation, we write  $B_n^{\bullet} = B_n^{\bullet}(\mathcal{T}_{\infty}^{(1)})$  and  $\partial^* B_n^{\bullet} = \partial^* B_n^{\bullet}(\mathcal{T}_{\infty}^{(1)})$  only in this subsection.

**Proposition 19.** *Let  $\varepsilon \in (0, 1)$  and  $\delta > 0$ . We can find  $\eta \in (0, 1/2)$  such that, for every sufficiently large  $n$ , the property*

$$(1 - \varepsilon)\mathbf{c}_0 \eta n \leq d_{\text{fpp}}(v, \partial^* B_{n - \lfloor \eta n \rfloor}^{\bullet}) \leq (1 + \varepsilon)\mathbf{c}_0 \eta n, \quad \forall v \in \partial^* B_n^{\bullet},$$

*holds with probability at least  $1 - \delta$ .*

*Proof.* Let us briefly outline the main steps of the proof. Recall that  $u_j^{(n)}$ , for  $0 \leq j \leq L_n - 1$ , are the vertices of  $\partial^* B_n^\bullet$  enumerated as explained before Proposition 17, and that we have extended the definition of  $u_j^{(n)}$  to all  $j \in \mathbb{Z}$  by periodicity. The first step of the proof is to use Proposition 15 to observe that an FPP shortest path from  $u_j^{(n)}$  (for some fixed  $j$ ) to  $\partial^* B_{n-\lfloor \eta n \rfloor}^\bullet$  that stays in  $B_n^\bullet$  cannot “turn around” the layer  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$ , and more precisely that it must stay in the region bounded by the left-most geodesics coming from  $u_{j-\lfloor cn^2 \rfloor}^{(n)}$  and  $u_{j+\lfloor cn^2 \rfloor}^{(n)}$  respectively, for some  $c > 0$ . Proposition 5 then allows us to compare the distribution of the trees of the skeleton of  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$  that code the latter region, with the distribution of independent Galton–Watson trees. This makes it possible to transfer the result known in the LHPT case (Proposition 18) to FPP distances in  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$ . Finally, to get uniformity in the starting vertex  $u_j^{(n)}$ , we use the coalescence property given in Proposition 17, which roughly speaking says that it is enough to consider a fixed number  $A$  (large but independent of  $n$ ) of values of  $j$ .

Let us turn to the details of the argument. As a consequence of the bounds of Lemma 4, we can fix  $a \in (0, 1/2)$  small enough (depending on  $\delta$ ) so that, for every  $n \geq 2$ , for every  $\eta \in (0, 1/2)$ , the event

$$\mathcal{E}_n(\eta) := \{\lfloor an^2 \rfloor + 1 \leq L_n \leq \lfloor a^{-1}n^2 \rfloor\} \cap \{\lfloor an^2 \rfloor + 1 \leq L_{n-\lfloor \eta n \rfloor} \leq \lfloor a^{-1}n^2 \rfloor\}$$

holds with probability at least  $1 - \delta/4$ .

For  $\eta \in (0, 1/2)$  and  $j \in \mathbb{Z}$ , let  $\mathcal{H}_{n,j}(\eta)$  be the intersection of  $\mathcal{E}_n(\eta)$  with the event where the leftmost geodesics starting respectively from  $u_{j-\lfloor an^2/4 \rfloor}^{(n)}$  and from  $u_{j+\lfloor an^2/4 \rfloor}^{(n)}$  do not coalesce before hitting  $B_{n-\lfloor \eta n \rfloor}^\bullet$ . By the considerations of the end of subsection 2.5, we know that, if  $\eta > 0$  is small enough, we have for every sufficiently large  $n$ , for every  $j \in \mathbb{Z}$ ,

$$\mathbb{P}(\mathcal{E}_n(\eta) \cap (\mathcal{H}_{n,j}(\eta))^c) \leq a^2 \delta / 80. \quad (54)$$

On the event  $\mathcal{H}_{n,j}(\eta)$ , we define  $\mathcal{G}_j^{(n)}(\eta)$  as the subregion of  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$  that contains  $u_j^{(n)}$  and is bounded on one side by the leftmost geodesic from  $u_{j-\lfloor an^2/4 \rfloor}^{(n)}$  and on the other side by the leftmost geodesic from  $u_{j+\lfloor an^2/4 \rfloor}^{(n)}$ . We also write  $\partial_\ell \mathcal{G}_j^{(n)}(\eta)$  for the part of the boundary of  $\mathcal{G}_j^{(n)}(\eta)$  that is contained in the union of the leftmost geodesics from  $u_{j-\lfloor an^2/4 \rfloor}^{(n)}$  and from  $u_{j+\lfloor an^2/4 \rfloor}^{(n)}$ .

Let  $\mathcal{A}_{n,j}(\eta)$  be the intersection of  $\mathcal{H}_{n,j}(\eta)$  with the event where, for some  $i$  with  $j - an^2/16 \leq i \leq j + an^2/16$ , there exists a path from  $u_i^{(n)}$  to  $\partial_\ell \mathcal{G}_j^{(n)}(\eta)$  that stays in  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$  and has length smaller than  $4\eta n / \kappa$  (recall that the weights of our first-passage percolation belong to  $[\kappa, 1]$ ). We claim that, by choosing  $\eta \in (0, \frac{1}{2})$  even smaller if necessary, we have also, for every sufficiently large  $n$  and for every  $j \in \mathbb{Z}$ ,

$$\mathbb{P}(\mathcal{A}_{n,j}(\eta)) \leq a^2 \delta / 80. \quad (55)$$

Let us prove this claim. Obviously it is enough to take  $j = 0$  (recall that  $u_0^{(n)}$  is chosen uniformly at random over  $\partial^* B_n^\bullet$ ). For every  $i \in \mathbb{Z}$ , write  $\mathcal{T}_i^{(n, \lfloor \eta n \rfloor)}$  for the tree in the skeleton of  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$  that is rooted at the edge between  $u_{i-1}^{(n)}$  and  $u_i^{(n)}$ . Then note that, on the event  $\mathcal{H}_{n,j}(\eta)$ , the region  $\mathcal{G}_j^{(n)}(\eta)$  is determined, as a planar map, by the trees  $\mathcal{T}_i^{(n, \lfloor \eta n \rfloor)}$  for  $-\lfloor an^2/4 \rfloor < i \leq \lfloor an^2/4 \rfloor$ , and by the Boltzmann distributed triangulations with a boundary used to fill in the slots associated with vertices of these trees at height strictly less than  $\lfloor \eta n \rfloor$ . Let  $\mathcal{F}$  be a forest with height  $\lfloor \eta n \rfloor$  such that the number of trees in  $\mathcal{F}$  and the number of vertices at generation  $\lfloor \eta n \rfloor$  both lie between  $\lfloor an^2 \rfloor + 1$  and  $\lfloor a^{-1}n^2 \rfloor$ . Proposition 5 shows that the probability of the event where the finite collection of random trees  $(\mathcal{T}_1^{(n, \lfloor \eta n \rfloor)}, \dots, \mathcal{T}_{L_n}^{(n, \lfloor \eta n \rfloor)})$  is equal to  $\mathcal{F}$  is bounded above, up to a multiplicative constant depending only on  $a$ , by the analogous probability for independent Galton–Watson trees with offspring distribution  $\theta$  truncated at generation  $\lfloor \eta n \rfloor$ . It follows that the probability of the event  $\mathcal{A}_{n,0}(\eta)$  can also be bounded by (a constant times)

the probability of the similar event in the LHPT model. More precisely,  $\mathbb{P}(\mathcal{A}_{n,0}(\eta))$  is bounded above, up to a multiplicative constant depending on  $a$ , by the probability that, in the half-plane model, there is a path going from  $(i, 0)$ , for some  $i$  such that  $-an^2/16 \leq i \leq an^2/16$ , to the leftmost geodesic from  $(\lfloor an^2/4 \rfloor, 0)$  (or from  $(-\lfloor an^2/4 \rfloor, 0)$ ) with length at most  $4\eta n/\kappa$ . If such a path exists, this implies that  $d_{\text{gr}}^{\mathcal{L}}((i, 0), (\lfloor an^2/4 \rfloor, 0)) \leq 8\eta n/\kappa$ . If we choose  $\eta$  small, the second assertion of Proposition 15 shows that the probability of the latter event can be made arbitrarily small, uniformly for all  $n \geq n_0$ , for some  $n_0$ . This gives our claim.

From now on, we fix  $\eta$  so that both (54) and (55) hold for  $n$  large. We set

$$\mathcal{B}_n(\eta) := \left( \bigcap_{k=0}^{\lfloor 9a^{-2} \rfloor} \mathcal{H}_{n,k\lfloor an^2/8 \rfloor}(\eta) \right) \cap \left( \bigcap_{k=0}^{\lfloor 9a^{-2} \rfloor} (\mathcal{A}_{n,k\lfloor an^2/8 \rfloor}(\eta))^c \right).$$

We note that, for  $n$  large,

$$\begin{aligned} \mathbb{P}(\mathcal{B}_n(\eta)^c) &\leq \mathbb{P}((\mathcal{E}_n(\eta))^c) + \sum_{k=0}^{\lfloor 9a^{-2} \rfloor} \mathbb{P}(\mathcal{E}_n(\eta) \cap (\mathcal{H}_{n,k\lfloor an^2/8 \rfloor}(\eta))^c) + \sum_{k=0}^{\lfloor 9a^{-2} \rfloor} \mathbb{P}(\mathcal{A}_{n,k\lfloor an^2/8 \rfloor}(\eta)) \\ &\leq \frac{\delta}{4} + (9a^{-2} + 1) \times \frac{a^2\delta}{80} + (9a^{-2} + 1) \times \frac{a^2\delta}{80} \\ &\leq \frac{\delta}{2}, \end{aligned}$$

using (54) and (55).

Next, consider, for every  $i \in \mathbb{Z}$ , the event

$$\mathcal{D}_i^{(n)} = \{d_{\text{fpp}}^{(n)}(u_i^{(n)}, \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet) \in [(1-\varepsilon)\mathbf{c}_0\eta n, (1+\varepsilon)\mathbf{c}_0\eta n]\},$$

where the FPP distance  $d_{\text{fpp}}^{(n)}$  is defined as  $d_{\text{fpp}}$ , but considering only paths that stay in  $B_n^\bullet$ .

Let  $k \in \{0, \dots, \lfloor 9a^{-2} \rfloor\}$  and assume that  $\mathcal{H}_{n,k\lfloor an^2/8 \rfloor}(\eta)$  and  $(\mathcal{A}_{n,k\lfloor an^2/8 \rfloor}(\eta))^c$  both hold. Then, for every  $i$  such that  $k\lfloor an^2/8 \rfloor - an^2/16 \leq i \leq k\lfloor an^2/8 \rfloor + an^2/16$ , the minimal FPP length of a path from  $u_i^{(n)}$  to  $\partial^* B_{n-\lfloor \eta n \rfloor}^\bullet$  that stays in  $B_n^\bullet \setminus B_{n-\lfloor \eta n \rfloor}^\bullet$  can be evaluated by considering only paths that stay in  $\mathcal{G}_{k\lfloor an^2/8 \rfloor}^{(n)}(\eta)$ . Indeed, the FPP length of a path that would hit  $\partial_t \mathcal{G}_{k\lfloor an^2/8 \rfloor}^{(n)}(\eta)$  is at least  $\kappa \times (4\eta n/\kappa) = 4\eta n$ , whereas the (graph distance) geodesic from  $u_i^{(n)}$  to  $\partial^* B_{n-\lfloor \eta n \rfloor}^\bullet$  has FPP length bounded above by  $\eta n$ . Hence, using Proposition 5, we have (provided that  $n$  is large enough), for every  $i$  such that  $k\lfloor an^2/8 \rfloor - an^2/16 \leq i \leq k\lfloor an^2/8 \rfloor + an^2/16$ ,

$$\mathbb{P}(\mathcal{B}_n(\eta) \cap (\mathcal{D}_i^{(n)})^c) \leq \mathbb{P}(\mathcal{H}_{n,k\lfloor an^2/8 \rfloor}(\eta) \cap (\mathcal{A}_{n,k\lfloor an^2/8 \rfloor}(\eta))^c \cap (\mathcal{D}_i^{(n)})^c) \leq C_1 \mathbb{P}(F_n) \quad (56)$$

where  $F_n$  denotes the event  $\{d_{\text{fpp}}^{\mathcal{L}}((0, 0), \mathcal{L}_{\lfloor \eta n \rfloor}) \notin [(1-\varepsilon)\mathbf{c}_0\eta n, (1+\varepsilon)\mathbf{c}_0\eta n]\}$  and  $C_1$  is a constant depending only on  $a$ . To derive the last bound, we note that, on the event  $\mathcal{H}_{n,k\lfloor an^2/8 \rfloor}(\eta)$ , the properties considered in the event  $(\mathcal{A}_{n,k\lfloor an^2/8 \rfloor}(\eta))^c \cap (\mathcal{D}_i^{(n)})^c$  can be expressed in terms of the trees  $\mathcal{T}_j^{(n, \lfloor \eta n \rfloor)}$  for  $k\lfloor an^2/8 \rfloor - \lfloor an^2/4 \rfloor < j \leq k\lfloor an^2/8 \rfloor + \lfloor an^2/4 \rfloor$  (together with the triangulations filling in the slots), and we again use the bound of Proposition 5.

Let  $A$  be the integer found in Proposition 17, where we replace  $\gamma$  by  $\varepsilon\mathbf{c}_0\eta/2$  and  $\delta$  by  $\delta/4$ . Using the conclusion of Proposition 17, we see that we have for all sufficiently large  $n$ ,

$$\begin{aligned} \mathcal{B}_n(\eta) \cap \{d_{\text{fpp}}^{(n)}(v, \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet) \notin [(1-2\varepsilon)\mathbf{c}_0\eta n, (1+2\varepsilon)\mathbf{c}_0\eta n], \text{ for some } v \in \partial^* B_n^\bullet\} \\ \subset \mathcal{B}_n(\eta) \cap \{d_{\text{fpp}}^{(n)}(u_{\lfloor jn^2/A \rfloor}^{(n)}, \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet) \notin [(1-\varepsilon)\mathbf{c}_0\eta n, (1+\varepsilon)\mathbf{c}_0\eta n], \text{ for some } 0 \leq j \leq \lfloor a^{-1}A \rfloor\}, \quad (57) \end{aligned}$$

except possibly on an event of probability at most  $\delta/4$ . The point is that if we assume that  $\mathcal{E}_n(\eta)$  holds (in particular if  $\mathcal{B}_n(\eta)$  holds) but discard the set of probability at most  $\delta/4$  considered in Proposition 17, any vertex of  $\partial^* B_n^\bullet$  will be at  $d_{\text{fpp}}^{(n)}$ -distance smaller than  $2 \times \varepsilon \mathbf{c}_0 \eta n/2$  from one of the vertices  $u_{\lfloor jn^2/A \rfloor}^{(n)}$ ,  $0 \leq j \leq \lfloor a^{-1}A \rfloor$ .

Next, using Proposition 18, we have, for all sufficiently large  $n$ ,

$$C_1 \mathbb{P}(F_n) \leq \frac{a\delta}{2(A+1)},$$

and it follows from (56) that  $\mathbb{P}(\mathcal{B}_n(\eta) \cap (\mathcal{D}_i^{(n)})^c) \leq \frac{a\delta}{2(A+1)}$  for every  $i \in \{0, 1, \dots, \lfloor a^{-1}n^2 \rfloor\}$  (observe that  $\lfloor 9a^{-2} \rfloor \times \lfloor an^2/8 \rfloor \geq \lfloor a^{-1}n^2 \rfloor$  if  $n$  is large).

Finally, the probability of the event in the right-hand side of (57) is bounded above for  $n$  large by

$$\sum_{j=0}^{\lfloor a^{-1}A \rfloor} \mathbb{P}\left(\mathcal{B}_n(\eta) \cap (\mathcal{D}_{\lfloor jn^2/A \rfloor}^{(n)})^c\right) \leq (\lfloor a^{-1}A \rfloor + 1) \times \frac{a\delta}{2(A+1)} \leq \frac{\delta}{2}.$$

Recalling that  $\mathbb{P}(\mathcal{B}_n(\eta)^c) \leq \delta/2$ , and using the last bound together with (57), we arrive at the bound

$$\mathbb{P}\left(d_{\text{fpp}}^{(n)}(v, \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet) \in [(1-2\varepsilon)\mathbf{c}_0 \lfloor \eta n \rfloor, (1+2\varepsilon)\mathbf{c}_0 \lfloor \eta n \rfloor], \text{ for every } v \in \partial^* B_n^\bullet\right) \geq 1 - \delta.$$

Now note that we can replace  $d_{\text{fpp}}^{(n)}$  by  $d_{\text{fpp}}$  in the last bound, since clearly  $d_{\text{fpp}} \leq d_{\text{fpp}}^{(n)}$ , and, on the other hand, it is also true that, for every  $v \in \partial^* B_n^\bullet$ ,

$$d_{\text{fpp}}(v, \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet) \geq \min_{v' \in \partial^* B_n^\bullet} d_{\text{fpp}}^{(n)}(v', \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet).$$

This completes the proof.  $\square$

**Proposition 20.** *For every  $\varepsilon \in (0, 1)$ ,*

$$\mathbb{P}\left((\mathbf{c}_0 - \varepsilon)n \leq d_{\text{fpp}}(\rho, v) \leq (\mathbf{c}_0 + \varepsilon)n, \text{ for every vertex } v \text{ in } \partial^* B_n^\bullet\right) \xrightarrow{n \rightarrow \infty} 1.$$

*Proof.* Fix  $\varepsilon \in (0, 1)$  and let  $\delta \in (0, \varepsilon/(4|\log(\varepsilon/16)|))$ . By Proposition 19, we can fix  $\eta \in (0, \frac{1}{4})$  such that, for every sufficiently large  $n$ , the event

$$\mathcal{G}_n := \left\{(\mathbf{c}_0 - \frac{\varepsilon}{2})\lfloor \eta n \rfloor \leq d_{\text{fpp}}(v, \partial^* B_{n-\lfloor \eta n \rfloor}^\bullet) \leq (\mathbf{c}_0 + \frac{\varepsilon}{2})\lfloor \eta n \rfloor, \forall v \in \partial^* B_n^\bullet\right\},$$

holds with probability at least  $1 - \delta^2$ .

Let  $n \geq 1$ . Set  $n_0 = n$ ,  $n_1 = n - \lfloor \eta n \rfloor$  and by induction  $n_i = n_{i-1} - \lfloor \eta n_{i-1} \rfloor$  for  $i \geq 1$ . Set

$$q = \left\lfloor \frac{\log(\varepsilon/16)}{\log(1-\eta)} \right\rfloor$$

so that we have  $n_q \leq \varepsilon n/4$  for  $n$  large enough. By our choice of  $\eta$ , we have  $\mathbb{E}\left[\sum_{j=0}^{q-1} \mathbf{1}_{\mathcal{G}_{n_j}^c}\right] \leq \delta^2 q$ , as soon as  $n$  is sufficiently large, and the Markov inequality gives

$$\mathbb{P}\left(\sum_{j=0}^{q-1} \mathbf{1}_{\mathcal{G}_{n_j}^c} > \delta q\right) \leq \delta.$$

In what follows, we argue on the event

$$\mathcal{H}_n := \left\{\sum_{j=0}^{q-1} \mathbf{1}_{\mathcal{G}_{n_j}^c} \leq \delta q\right\}.$$

Let  $v \in \partial^* B_n^\bullet$ . We construct inductively a finite sequence  $(v_{(j)})_{0 \leq j \leq q}$ , such that  $v_{(j)} \in \partial^* B_{n_j}^\bullet$ , for every  $0 \leq j \leq q$ . We start with  $v_{(0)} = v$ . Then, if we have constructed  $v_{(0)}, \dots, v_{(j)}$  for some  $0 \leq j < q$ , we define  $v_{(j+1)}$  as follows. If the event  $\mathcal{G}_{n_j}$  holds, we let  $v_{(j+1)}$  be any point in  $\partial^* B_{n_{j+1}}^\bullet$  such that  $d_{\text{fpp}}(v_{(j)}, v_{(j+1)}) = d_{\text{fpp}}(v_{(j)}, \partial^* B_{n_{j+1}}^\bullet)$ . Otherwise, we choose  $v_{(j+1)} \in \partial^* B_{n_{j+1}}^\bullet$  such that  $d_{\text{gr}}(v_{(j)}, v_{(j+1)}) = n_j - n_{j+1}$ . We note that  $d_{\text{fpp}}(\rho, v_{(q)}) \leq d_{\text{gr}}(\rho, v_{(q)}) = n_q \leq \varepsilon n/4$ . Hence, for  $n$  large enough, we have on the event  $\mathcal{H}_n$ ,

$$\begin{aligned} d_{\text{fpp}}(\rho, v) &\leq \sum_{j=0}^{q-1} d_{\text{fpp}}(v_{(j)}, v_{(j+1)}) + \frac{\varepsilon n}{4} \\ &\leq (\mathbf{c}_0 + \frac{\varepsilon}{2}) \sum_{j=0}^{q-1} (n_j - n_{j+1}) + \delta q \max_{0 \leq i < q} \{n_i - n_{i+1}\} + \frac{\varepsilon n}{4} \\ &\leq (\mathbf{c}_0 + \frac{\varepsilon}{2}) n + \delta q \eta n + \frac{\varepsilon n}{4} \\ &\leq (\mathbf{c}_0 + \varepsilon) n, \end{aligned}$$

where we used in the last line the fact that  $\eta \leq |\log(1 - \eta)|$  for  $\eta \in (0, 1)$  to get that  $\delta q \eta \leq \varepsilon/4$ .

On the other hand, take any path  $\omega$  from  $v$  to  $\rho$ , and for every integer  $j \in \{0, 1, \dots, q\}$ , write  $w_{(j)}$  for the last point of  $\omega$  that belongs to  $\partial^* B_{n_j}^\bullet$ . Then, for  $n$  large enough, on the event  $\mathcal{H}_n$ , the weight of the path  $\omega$  is bounded below by

$$\begin{aligned} \sum_{j=0}^{q-1} d_{\text{fpp}}(w_{(j)}, \partial^* B_{n_{j+1}}^\bullet) &\geq (\mathbf{c}_0 - \frac{\varepsilon}{2})(n_0 - n_q) - \delta q \max_{0 \leq i < q} \{n_i - n_{i+1}\} \\ &\geq n(\mathbf{c}_0 - \frac{\varepsilon}{2})(1 - \frac{\varepsilon}{4}) - \delta q \eta n \\ &\geq (\mathbf{c}_0 - \varepsilon) n, \end{aligned}$$

since we have  $n_q \leq \varepsilon n/4$ ,  $c_0 \leq 1$  and  $\delta q \eta \leq \varepsilon/4$ . This implies that, on the event  $\mathcal{H}_n$ , we have

$$d_{\text{fpp}}(\rho, v) \geq (\mathbf{c}_0 - \varepsilon) n.$$

Since we have  $\mathbb{P}((\mathcal{H}_n)^c) \leq \delta$ , for all sufficiently large  $n$ , where  $\delta$  can be taken arbitrarily small, this completes the proof.  $\square$

## 6 First-passage percolation on finite triangulations

For every  $n \geq 1$ , let  $\mathcal{T}_n^{(1)}$  be uniformly distributed over  $\mathbb{T}_{n,1}$ . Recall from Fig. 2 that we can transform  $\mathcal{T}_n^{(1)}$  into a uniform rooted plane triangulation with  $n+1$  vertices, which is denoted by  $\mathcal{T}_n$ . We write  $\rho_n$  for the root vertex of  $\mathcal{T}_n^{(1)}$ . We assign i.i.d. weights to the edges of  $\mathcal{T}_n^{(1)}$ , with the same distribution as in the last section, and we write  $d_{\text{fpp}}$  for the associated first-passage percolation distance on the vertex set  $V(\mathcal{T}_n^{(1)})$ . We also keep the notation  $\mathbf{c}_0$  for the constant in Proposition 18.

**Proposition 21.** *Let  $o_n$  be a uniformly distributed inner vertex of  $\mathcal{T}_n^{(1)}$ . Then, for every  $\varepsilon > 0$ ,*

$$\mathbb{P}\left(\left|d_{\text{fpp}}(\rho_n, o_n) - \mathbf{c}_0 d_{\text{gr}}(\rho_n, o_n)\right| > \varepsilon n^{1/4}\right) \xrightarrow{n \rightarrow \infty} 0.$$

The proof relies on certain absolute continuity relations between finite triangulations and the UIPT, which are similar to [18, Section 4.3]. We start with a preliminary lemma. Recall our notation  $\mathbb{C}_{1,r}$  for

the set of all triangulations of the cylinder of height  $r$  with bottom cycle of length 1. If  $t \in \mathbb{C}_{1,r}$ , we denote the total number of vertices of  $t$  by  $N(t) + 1$ .

We write  $\overline{\mathcal{T}}_n^{(1)}$  for the triangulation  $\mathcal{T}_n^{(1)}$  given with the distinguished vertex  $o_n$ . The hull  $B_r^\bullet(\overline{\mathcal{T}}_n^{(1)})$  makes sense provided that  $d_{\text{gr}}(\rho_n, o_n) > r$ , and otherwise we let  $B_r^\bullet(\overline{\mathcal{T}}_n^{(1)})$  be the edge-triangulation.

**Lemma 22.** *There exists a constant  $\bar{c}$  such that, for every  $n \geq 1$ , for every  $r \geq 1$  and every  $t \in \mathbb{C}_{1,r}$  such that  $n > N(t)$ ,*

$$\mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) = t) \leq \bar{c} \left( \frac{n}{n - N(t)} \right)^{3/2} \mathbb{P}(B_r^\bullet(\mathcal{T}_\infty^{(1)}) = t). \quad (58)$$

*Proof.* Fix  $r \geq 1$  and  $t \in \mathbb{C}_{1,r}$  and write  $N = N(t)$  to simplify notation. As a consequence of (13) and of the fact that  $\mathcal{T}_\infty^{(1)}$  is the local limit of  $\mathcal{T}_n^{(1)}$  as  $n \rightarrow \infty$  (see the end of subsection 2.4), we know that

$$\mathbb{P}(B_r^\bullet(\mathcal{T}_\infty^{(1)}) = t) = \frac{C(p)}{C(1)} (12\sqrt{3})^{-N}, \quad (59)$$

where  $p = |\partial^* t|$  is the length of the top cycle of  $t$ . On the other hand, (12) gives the explicit formula

$$\mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) = t) = \frac{n - N}{n} \frac{\#\mathbb{T}_{n-N,p}}{\#\mathbb{T}_{n,1}}$$

from which, using Lemma 1 and the asymptotics (6) (with  $p = 1$ ), we derive the bound

$$\mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) = t) \leq c^* C(p) \left( \frac{n}{n - N} \right)^{3/2} (12\sqrt{3})^{-N},$$

with some constant  $c^*$ . By comparing the last bound with (59), we get the desired result.  $\square$

*Proof of Proposition 21.* We fix  $\varepsilon > 0$  and  $\nu \in (0, 1)$ . We will prove that for all sufficiently large  $n$  we have

$$\mathbb{P} \left( \left| \frac{d_{\text{fpp}}(\rho_n, o_n)}{d_{\text{gr}}(\rho_n, o_n)} - \mathbf{c}_0 \right| > 2\varepsilon \right) < \nu.$$

Since the Gromov–Hausdorff convergence of rescaled triangulations to the Brownian map [27] (see Theorem 6 in Appendix A1 below) implies that the sequence  $n^{-1/4} d_{\text{gr}}(\rho_n, o_n)$  is bounded in probability, the statement of the proposition follows.

Let  $r \geq 1$ . For every  $t \in \mathbb{C}_{1,r}$ , we can equip the edges of  $t$  with i.i.d. weights distributed as previously, and then consider the associated first-passage percolation distance  $d_{\text{fpp}}$ . We let  $a_\varepsilon(t)$  be the random variable defined by  $a_\varepsilon(t) = 1$  if

$$\sup_{x \in \partial^* t} \left| \frac{d_{\text{fpp}}(\rho, x)}{d_{\text{gr}}(\rho, x)} - \mathbf{c}_0 \right| \geq \varepsilon$$

and  $a_\varepsilon(t) = 0$  otherwise (here  $\rho$  stands for the root vertex of  $t$  and we recall that  $\partial^* t$  is the top cycle of  $t$ ). By convention we also define  $a_\varepsilon(t_0) = 0$ , when  $t_0$  is the edge-triangulation.

Let  $b \in (0, 1)$ . We observe that

$$\begin{aligned} \mathbb{P} \left( \{a_\varepsilon(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)})) = 1\} \cap \{\#B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) \leq (1-b)n\} \right) &= \sum_{t \in \mathbb{C}_{1,r}, N(t)+1 \leq (1-b)n} \mathbb{P}(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) = t) \mathbb{P}(a_\varepsilon(t) = 1) \\ &\leq \bar{c} b^{-3/2} \sum_{t \in \mathbb{C}_{1,r}} \mathbb{P}(B_r^\bullet(\mathcal{T}_\infty^{(1)}) = t) \mathbb{P}(a_\varepsilon(t) = 1), \end{aligned}$$

using the bound (58). It follows that

$$\mathbb{P} \left( \{a_\varepsilon(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)})) = 1\} \cap \{\#B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) \leq (1-b)n\} \right) \leq \bar{c} b^{-3/2} \mathbb{P}(a_\varepsilon(B_r^\bullet(\mathcal{T}_\infty^{(1)})) = 1).$$

Using Proposition 20, we now get

$$\lim_{r \rightarrow \infty} \left( \sup_{n \geq 1} \mathbb{P} \left( \{a_\varepsilon(B_r^\bullet(\overline{\mathcal{T}}_n^{(1)})) = 1\} \cap \{\#B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) < (1-b)n\} \right) \right) = 0. \quad (60)$$

Let us fix  $0 < \alpha < \beta < \gamma$ . Write  $\mathfrak{B}_r(\mathcal{T}_n^{(1)}, o_n)$  for the ball of radius  $r$  centered at  $o_n$  in  $\mathcal{T}_n^{(1)}$ . For every  $n \geq 1$ , consider the event

$$D_{\beta, \gamma, n} := \{\beta n^{1/4} < d_{\text{gr}}(\rho_n, o_n) \leq \gamma n^{1/4}\}.$$

For future use, we note that  $B_{\lfloor \alpha n^{1/4} \rfloor}^\bullet(\overline{\mathcal{T}}_n^{(1)})$  is nontrivial on the event  $D_{\beta, \gamma, n}$ . We then observe that

$$\left( D_{\beta, \gamma, n} \cap \{\# \mathfrak{B}_{\lfloor (\beta - \alpha) n^{1/4} \rfloor}(\mathcal{T}_n^{(1)}, o_n) > bn\} \right) \subset \{\# B_{\lfloor \alpha n^{1/4} \rfloor}^\bullet(\overline{\mathcal{T}}_n^{(1)}) \leq (1-b)n\},$$

simply because if  $D_{\beta, \gamma, n}$  holds, the whole ball  $\mathfrak{B}_{(\beta - \alpha) n^{1/4}}(\mathcal{T}_n^{(1)}, o_n)$  is contained in the complement of the hull  $B_{\lfloor \alpha n^{1/4} \rfloor}^\bullet(\overline{\mathcal{T}}_n^{(1)})$ . It then follows from (60) that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \{a_\varepsilon(B_{\lfloor \alpha n^{1/4} \rfloor}^\bullet(\overline{\mathcal{T}}_n^{(1)})) = 1\} \cap D_{\beta, \gamma, n} \cap \{\# \mathfrak{B}_{\lfloor (\beta - \alpha) n^{1/4} \rfloor}(\mathcal{T}_n^{(1)}, o_n) > bn\} \right) = 0. \quad (61)$$

On the other hand, given any  $y < 1$ , we can choose  $b \in (0, 1)$  such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\# \mathfrak{B}_{\lfloor (\beta - \alpha) n^{1/4} \rfloor}(\mathcal{T}_n^{(1)}, o_n) > bn) \geq y. \quad (62)$$

This follows from the relation between  $\mathcal{T}_n^{(1)}$  and  $\mathcal{T}_n$ , and the well-known convergence in distribution of the rescaled profile of distances from a vertex chosen uniformly at random in  $\mathcal{T}_n$  toward a random measure that gives positive mass to every neighborhood of 0: See Theorem 1(iii) in [31]. Since (62) holds for  $y$  arbitrarily close to 1, we deduce from (61) that we have also

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \{a_\varepsilon(B_{\lfloor \alpha n^{1/4} \rfloor}^\bullet(\overline{\mathcal{T}}_n^{(1)})) = 1\} \cap D_{\beta, \gamma, n} \right) = 0. \quad (63)$$

To complete the argument, choose  $\delta \in (0, 1/2)$  such that  $2\delta < \varepsilon$ . We will apply (63) to

$$\alpha_j = j\delta^2, \quad \beta_j = (j+1)\delta^2, \quad \gamma_j = (j+2)\delta^2$$

for integers  $j$  such that  $\lfloor \delta^{-1} \rfloor < j \leq \lfloor \delta^{-3} \rfloor$ . We observe that

$$\mathbb{P} \left( \bigcup_{j=\lfloor \delta^{-1} \rfloor + 1}^{\lfloor \delta^{-3} \rfloor} D_{\beta_j, \gamma_j, n} \right) = \mathbb{P} \left( (\lfloor \delta^{-1} \rfloor + 2)\delta^2 n^{1/4} < d_{\text{gr}}(\rho_n, o_n) \leq (\lfloor \delta^{-3} \rfloor + 2)\delta^2 n^{1/4} \right).$$

Now recall that  $n^{-1/4} d_{\text{gr}}(\rho_n, o_n)$  converges in distribution to a positive random variable (see e.g. [34, Theorem 1.2(iii)]). Hence, by choosing  $\delta$  smaller if needed, it follows from the previous display that we have for all sufficiently large  $n$ ,

$$\mathbb{P} \left( \bigcup_{j=\lfloor \delta^{-1} \rfloor + 1}^{\lfloor \delta^{-3} \rfloor} D_{\beta_j, \gamma_j, n} \right) \geq 1 - \frac{\nu}{2}.$$

If we combine this with (63) (applied with  $\alpha = \alpha_j, \beta = \beta_j, \gamma = \gamma_j$  for the relevant values of  $j$ ), we get that, for  $n$  large enough,

$$\mathbb{P} \left( \bigcup_{j=\lfloor \delta^{-1} \rfloor + 1}^{\lfloor \delta^{-3} \rfloor} \left( \{a_\varepsilon(B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\overline{\mathcal{T}}_n^{(1)})) = 0\} \cap D_{\beta_j, \gamma_j, n} \right) \right) \geq 1 - \nu.$$

To complete the proof, we just need to verify that we have

$$\left| \frac{d_{\text{fpp}}(\rho_n, o_n)}{d_{\text{gr}}(\rho_n, o_n)} - \mathbf{c}_0 \right| \leq 2\varepsilon$$

on the event whose probability is considered in the previous display. Indeed, suppose that, for some  $j \in \{\lfloor \delta^{-1} \rfloor + 1, \dots, \lfloor \delta^{-3} \rfloor\}$ , the event  $\{a_\varepsilon(B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\bar{\mathcal{T}}_n^{(1)})) = 0\} \cap D_{\beta_j, \gamma_j, n}$  holds. Then, clearly,

$$d_{\text{fpp}}(\rho_n, o_n) \geq \min \{d_{\text{fpp}}(\rho_n, x) : x \in \partial^* B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\bar{\mathcal{T}}_n^{(1)})\} \geq (\mathbf{c}_0 - \varepsilon) \lfloor \alpha_j n^{1/4} \rfloor,$$

and it follows that

$$\frac{d_{\text{fpp}}(\rho_n, o_n)}{d_{\text{gr}}(\rho_n, o_n)} \geq \frac{(\mathbf{c}_0 - \varepsilon) \lfloor \alpha_j n^{1/4} \rfloor}{\gamma_j n^{1/4}} \geq \mathbf{c}_0 - 2\varepsilon.$$

using the fact that  $\frac{\alpha_j}{\gamma_j} = \frac{j}{j+2} \geq 1 - \frac{2}{j} > 1 - 2\delta > 1 - \varepsilon$ . On the other hand, still on the event  $\{a_\varepsilon(B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\bar{\mathcal{T}}_n^{(1)})) = 0\} \cap D_{\beta_j, \gamma_j, n}$ , we have

$$\begin{aligned} d_{\text{fpp}}(\rho_n, o_n) &\leq \left( \max \{d_{\text{fpp}}(\rho_n, x) : x \in \partial^* B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\bar{\mathcal{T}}_n^{(1)})\} \right) + (\lfloor \gamma_j n^{1/4} \rfloor - \lfloor \alpha_j n^{1/4} \rfloor) \\ &\leq (\mathbf{c}_0 + \varepsilon) \lfloor \alpha_j n^{1/4} \rfloor + (\lfloor \gamma_j n^{1/4} \rfloor - \lfloor \alpha_j n^{1/4} \rfloor) \end{aligned}$$

which implies

$$\frac{d_{\text{fpp}}(\rho_n, o_n)}{d_{\text{gr}}(\rho_n, o_n)} \leq \frac{(\mathbf{c}_0 + \varepsilon) \lfloor \alpha_j n^{1/4} \rfloor + (\lfloor \gamma_j n^{1/4} \rfloor - \lfloor \alpha_j n^{1/4} \rfloor)}{\beta_j n^{1/4}} \leq \mathbf{c}_0 + 2\varepsilon.$$

This completes the proof.  $\square$

In the next theorem, we deal with the uniform rooted plane triangulation  $\mathcal{T}_n$  with  $n+1$  vertices. We equip the vertex set of  $\mathcal{T}_n$  with the first-passage percolation distance  $d_{\text{fpp}}$  defined as previously.

**Theorem 1.** *For every  $\varepsilon > 0$ , we have*

$$\mathbb{P} \left( \sup_{x, y \in \mathcal{V}(\mathcal{T}_n)} |d_{\text{fpp}}(x, y) - \mathbf{c}_0 d_{\text{gr}}(x, y)| > \varepsilon n^{1/4} \right) \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* As mentioned above, we may assume that  $\mathcal{T}_n$  and  $\mathcal{T}_n^{(1)}$  are linked via the transformation of Fig. 2. Then  $\mathcal{V}(\mathcal{T}_n^{(1)}) = \mathcal{V}(\mathcal{T}_n)$  and the graph distances are the same in  $\mathcal{V}(\mathcal{T}_n^{(1)})$  and in  $\mathcal{V}(\mathcal{T}_n)$ . The root vertex  $\rho_n$  is also the same in  $\mathcal{T}_n$  and in  $\mathcal{T}_n^{(1)}$ . Furthermore, if  $o_n$  stands for a uniformly distributed inner vertex of  $\mathcal{T}_n^{(1)}$ , as in Proposition 21, we can couple  $o_n$  with a uniformly distributed vertex  $o'_n$  of  $\mathcal{V}(\mathcal{T}_n)$ , so that  $\mathbb{P}(o_n = o'_n) = n/(n+1)$ . Finally, we can assume that the FPP weights are the same for all edges shared by  $\mathcal{T}_n$  and  $\mathcal{T}_n^{(1)}$  (that is, for all edges except for those involved in the transformation of Fig. 2). It then follows from Proposition 21 that we have

$$\mathbb{P} \left( |d_{\text{fpp}}(\rho_n, o'_n) - \mathbf{c}_0 d_{\text{gr}}(\rho_n, o'_n)| > \varepsilon n^{1/4} \right) \xrightarrow{n \rightarrow \infty} 0, \quad (64)$$

where the graph and FPP distances refer to  $\mathcal{T}_n$ . Indeed, on the event  $\{o_n = o'_n\}$ , the graph distance  $d_{\text{gr}}(\rho_n, o'_n)$  (in  $\mathcal{T}_n$ ) is the same as the graph distance  $d_{\text{gr}}(\rho_n, o_n)$  (in  $\mathcal{T}_n^{(1)}$ ), and the FPP distance  $d_{\text{fpp}}(\rho_n, o'_n)$  (in  $\mathcal{T}_n$ ) may only differ from the FPP distance  $d_{\text{fpp}}(\rho_n, o_n)$  (in  $\mathcal{T}_n^{(1)}$ ) by a quantity bounded in probability.

Write  $\bar{\mathcal{T}}_n$  for  $\mathcal{T}_n$  pointed at  $o'_n$ . Conditionally on  $\bar{\mathcal{T}}_n$ , choose an oriented edge  $e_n$  of  $\mathcal{T}_n$  uniformly at random. Then  $\bar{\mathcal{T}}_n$  re-rooted at  $e_n$  (and with the same distinguished vertex  $o'_n$ ) has the same distribution as  $\bar{\mathcal{T}}_n$ . It follows that (64) still holds if  $\rho_n$  is replaced by the root  $\rho'_n$  of  $e_n$ .



Let  $\vec{E}(\mathcal{T}_n)$  be the set of all oriented edges of  $\mathcal{T}_n$ , and for  $e \in \vec{E}(\mathcal{T}_n)$  let  $e_*$  denote the initial vertex of  $e$ . Since  $\#\vec{E}(\mathcal{T}_n) = 6(n-1)$  and  $\#\mathcal{V}(\mathcal{T}_n) = n+1$ , the probability in (64) (with  $\rho_n$  replaced by  $\rho'_n$ ) can be rewritten as

$$\mathbb{E} \left[ \frac{1}{n+1} \frac{1}{6(n-1)} \sum_{v \in \mathcal{V}(\mathcal{T}_n)} \sum_{e \in \vec{E}(\mathcal{T}_n)} \mathbf{1}_{\{|d_{\text{fpp}}(e_*, v) - \mathbf{c}_0 d_{\text{gr}}(e_*, v)| > \varepsilon n^{1/4}\}} \right]$$

and, since any vertex of  $\mathcal{T}_n$  can be written as  $e_*$  for (at least) one choice of  $e$ , this is bounded below by

$$\frac{1}{6(n+1)^2} \mathbb{E} \left[ \sum_{v \in \mathcal{V}(\mathcal{T}_n)} \sum_{\tilde{v} \in \mathcal{V}(\mathcal{T}_n)} \mathbf{1}_{\{|d_{\text{fpp}}(v, \tilde{v}) - \mathbf{c}_0 d_{\text{gr}}(v, \tilde{v})| > \varepsilon n^{1/4}\}} \right].$$

Therefore the quantity in the last display also tends to 0 as  $n \rightarrow \infty$ . This is just saying that, if  $o''_n$  is another vertex of  $\mathcal{V}(\mathcal{T}_n)$ , which conditionally on  $\bar{\mathcal{T}}_n$  is uniformly distributed over  $\mathcal{V}(\mathcal{T}_n)$ , we have also, for every  $\varepsilon > 0$ ,

$$\mathbb{P} \left( |d_{\text{fpp}}(o'_n, o''_n) - \mathbf{c}_0 d_{\text{gr}}(o'_n, o''_n)| > \varepsilon n^{1/4} \right) \xrightarrow{n \rightarrow \infty} 0. \quad (65)$$

Consider then, conditionally on  $\mathcal{T}_n$ , a sequence  $(o_n^i)_{i \geq 1}$  of vertices chosen independently at random uniformly over  $\mathcal{V}(\mathcal{T}_n)$ . Given any fixed  $\delta > 0$ , we can choose an integer  $N \geq 1$  large enough such that, for every  $n$ ,

$$\mathbb{P} \left( \sup_{x \in \mathcal{V}(\mathcal{T}_n)} \left( \inf_{1 \leq j \leq N} d_{\text{gr}}(x, o_n^j) \right) < \varepsilon n^{1/4} \right) > 1 - \delta. \quad (66)$$

This essentially follows from the convergence of rescaled triangulations to the Brownian map obtained in [27]. We provide a detailed proof of (66) in Appendix A1.

Once  $N$  is fixed, we deduce from (65) that we have also, for all sufficiently large  $n$ ,

$$\mathbb{P} \left( \bigcap_{1 \leq i \leq j \leq N} \{|d_{\text{fpp}}(o_n^i, o_n^j) - \mathbf{c}_0 d_{\text{gr}}(o_n^i, o_n^j)| \leq \varepsilon n^{1/4}\} \right) > 1 - \delta.$$

To complete the proof, just observe that

$$\begin{aligned} \sup_{x, y \in \mathcal{V}(\mathcal{T}_n)} |d_{\text{fpp}}(x, y) - \mathbf{c}_0 d_{\text{gr}}(x, y)| &\leq \sup_{i, j \in \{1, \dots, N\}} |d_{\text{fpp}}(o_n^i, o_n^j) - \mathbf{c}_0 d_{\text{gr}}(o_n^i, o_n^j)| \\ &\quad + 4 \sup_{x \in \mathcal{V}(\mathcal{T}_n)} \left( \inf_{1 \leq j \leq N} d_{\text{gr}}(x, o_n^j) \right) \end{aligned}$$

and the preceding two displays show that the right-hand side is bounded above by  $5\varepsilon n^{1/4}$  outside a set of probability at most  $2\delta$ , for all sufficiently large  $n$ .  $\square$

We now return to the UIPT  $\mathcal{T}_\infty$ , which we equip with the first-passage percolation distance  $d_{\text{fpp}}$ . We write  $B_r^{\text{fpp}}(\mathcal{T}_\infty)$  for the ball of radius  $r$  in  $\mathcal{T}_\infty$  for the first-passage percolation distance: This ball may be defined as the union of all faces of the UIPT that are incident to a vertex at  $d_{\text{fpp}}$ -distance strictly less than  $r$  from the root.

**Theorem 2.** *Let  $\varepsilon \in (0, 1)$ . We have*

$$\lim_{r \rightarrow \infty} \mathbb{P} \left( \sup_{x, y \in \mathcal{V}(B_r(\mathcal{T}_\infty))} |d_{\text{fpp}}(x, y) - \mathbf{c}_0 d_{\text{gr}}(x, y)| > \varepsilon r \right) = 0.$$

Consequently,

$$\mathbb{P} \left( B_{(1-\varepsilon)r/\mathbf{c}_0}(\mathcal{T}_\infty) \subset B_r^{\text{fpp}}(\mathcal{T}_\infty) \subset B_{(1+\varepsilon)r/\mathbf{c}_0}(\mathcal{T}_\infty) \right) \xrightarrow{r \rightarrow \infty} 1.$$

*Proof.* The second part of the theorem is an easy consequence of the first one, and so we concentrate on the first assertion. By the same arguments as in the beginning of the proof of Theorem 1, it is enough to prove the desired result with  $\mathcal{T}_\infty$  replaced by  $\mathcal{T}_\infty^{(1)}$ . We fix  $\delta > 0$ , which can be taken arbitrarily small. Consider first an arbitrary (deterministic) rooted planar map  $m$  with root vertex  $\rho$ . We equip the vertex set  $V(m)$  with the graph distance  $d_{\text{gr}}$  and with the (random) first-passage percolation distance  $d_{\text{fpp}}$ . For every integer  $r > 0$ , we say that  $m \in A_r^{(\delta)}$  if the property

$$|d_{\text{fpp}}(x, y) - c_0 d_{\text{gr}}(x, y)| \leq \varepsilon r \text{ for all } x, y \in V(m) \text{ such that } d_{\text{gr}}(\rho, x) \vee d_{\text{gr}}(\rho, y) \leq r,$$

holds with probability at least  $1 - \delta$ .

In order to prove the first assertion of the theorem, it is enough to verify that, for every fixed integer  $K \geq 1$ , we have for all sufficiently large integers  $r$ ,

$$\mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) \in A_r^{(\delta)}) \geq 1 - \delta.$$

The point is that if  $K$  is chosen sufficiently large, if  $x$  and  $y$  are two vertices of  $B_r(\mathcal{T}_\infty^{(1)})$ , the first-passage percolation distance (resp. the graph distance) between  $x$  and  $y$  in the graph  $\mathcal{T}_\infty^{(1)}$  coincides with the first-passage percolation distance (resp. the graph distance) between  $x$  and  $y$  in the graph  $B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)})$ .

So let us fix  $K \geq 1$ . Recall our notation  $1 + N(t)$  for the total number of vertices of  $t \in \mathbb{C}_{1,r}$ . We first observe that, by [17, Theorem 2], we can choose two positive integers  $\alpha > 1$  and  $\beta > 1$  such that, if  $D_r := \{t \in \mathbb{C}_{1,Kr} : N(t) \geq \alpha r^4 \text{ or } N(t) \leq \alpha^{-1} r^4 \text{ or } |\partial^* t| \geq \beta r^2\}$ , we have

$$\mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) \in D_r) \leq \frac{\delta}{2}.$$

Note that Theorem 2 in [17] deals with the type II UIPT, but the last section of [17] explains that a similar result holds for the type I triangulations that we consider here.

Then, if  $t \in \mathbb{C}_{1,Kr} \setminus D_r$ , it follows from formula (12), using both assertions of Lemma 1, that the quantity  $\mathbb{P}(B_{Kr}^\bullet(\overline{\mathcal{T}}_{2\alpha r^4}^{(1)}) = t)$  is bounded below by

$$\frac{c' C(|\partial^* t|) (2\alpha r^4 - N(t))^{-3/2} (12\sqrt{3})^{2\alpha r^4 - N(t)}}{c C(1) (2\alpha r^4)^{-3/2} (12\sqrt{3})^{2\alpha r^4}} \geq \frac{c'}{c} \frac{C(|\partial^* t|)}{C(1)} (12\sqrt{3})^{-N(t)} = \frac{c'}{c} \mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) = t),$$

where the last equality is (59). Summarizing, we have obtained the existence of a constant  $c'' > 0$  such that, for every  $t \in \mathbb{C}_{1,Kr} \setminus D_r$ ,

$$\mathbb{P}(B_{Kr}^\bullet(\overline{\mathcal{T}}_{2\alpha r^4}^{(1)}) = t) \geq c'' \mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) = t).$$

It follows that

$$\begin{aligned} \mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) \notin A_r^{(\delta)}) &\leq \mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) \in D_r) + \sum_{t \in (A_r^{(\delta)})^c \cap (\mathbb{C}_{1,Kr} \setminus D_r)} \mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) = t) \\ &\leq \frac{\delta}{2} + (c'')^{-1} \sum_{t \in (A_r^{(\delta)})^c \cap (\mathbb{C}_{1,Kr} \setminus D_r)} \mathbb{P}(B_{Kr}^\bullet(\overline{\mathcal{T}}_{2\alpha r^4}^{(1)}) = t) \\ &\leq \frac{\delta}{2} + (c'')^{-1} \mathbb{P}(B_{Kr}^\bullet(\overline{\mathcal{T}}_{2\alpha r^4}^{(1)}) \notin A_r^{(\delta)}). \end{aligned}$$

However, Theorem 1, or more precisely the equivalent statement for triangulations of the 1-gon, tells us that  $\mathbb{P}(B_{Kr}^\bullet(\overline{\mathcal{T}}_{2\alpha r^4}^{(1)}) \notin A_r^{(\delta)})$  tends to 0 as  $r \rightarrow \infty$ . We thus obtain that  $\mathbb{P}(B_{Kr}^\bullet(\mathcal{T}_\infty^{(1)}) \notin A_r^{(\delta)}) < \delta$  for all sufficiently large  $r$ , which was the desired result.  $\square$

## 7 Dual and Eden distances

In this section, we discuss variants of Theorems 1 and 2 that hold for particular choices of distances on the dual graph associated with finite triangulations, or with the UIPT. These choices correspond to the distances  $d_{\text{gr}}^\dagger$  or  $d_{\text{Eden}}^\dagger$  introduced in cases 1. and 2. discussed in the Introduction. The main technical difference when dealing with these distances on the dual graph comes from the lack of an priori bound (both from above and from below) for the modified distance by a multiple of the graph distance.

### 7.1 Statement of the results and identification of the constants

Recall our notation  $\mathcal{T}_n$  for a uniformly distributed rooted plane triangulation with  $n + 1$  vertices, and  $F(\mathcal{T}_n)$  for the set of all faces of  $\mathcal{T}_n$ . The dual graph distance on  $F(\mathcal{T}_n)$  is denoted by  $d_{\text{gr}}^\dagger$ . As in the Introduction, we also define the Eden distance  $d_{\text{Eden}}^\dagger$  on  $F(\mathcal{T}_n)$  by assigning independently to every dual edge an exponential weight with parameter 1, and considering the associated first-passage percolation distance.

We now state our analog of Theorem 1 for these distances.

**Theorem 3.** *There exist two constants  $\mathbf{c}_1, \mathbf{c}_2 \in (0, \infty)$  such that for every  $\varepsilon > 0$ , we have*

$$\mathbb{P} \left( \sup_{\substack{x, y \in V(\mathcal{T}_n), f, g \in F(\mathcal{T}_n) \\ x \triangleleft f, y \triangleleft g}} |d_{\text{gr}}^\dagger(f, g) - \mathbf{c}_1 d_{\text{gr}}(x, y)| > \varepsilon n^{1/4} \right) \xrightarrow{n \rightarrow \infty} 0,$$

$$\mathbb{P} \left( \sup_{\substack{x, y \in V(\mathcal{T}_n), f, g \in F(\mathcal{T}_n) \\ x \triangleleft f, y \triangleleft g}} |d_{\text{Eden}}^\dagger(f, g) - \mathbf{c}_2 d_{\text{gr}}(x, y)| > \varepsilon n^{1/4} \right) \xrightarrow{n \rightarrow \infty} 0,$$

where we recall that the notation  $x \triangleleft f$  means that the vertex  $x$  is incident to the face  $f$ .

Combining the above theorem with Theorem 1 and the known convergence of rescaled triangulations towards the Brownian map we obtain the following joint convergences. If  $(E, d)$  is a metric space and  $\alpha > 0$ , we use the notation  $\alpha \cdot (E, d) = (E, \alpha d)$ .

**Corollary 23.** *Let  $(\mathbf{m}_\infty, D^*)$  be the Brownian map. The following convergences in distribution*

$$\begin{aligned} 3^{1/4} n^{-1/4} \cdot (V(\mathcal{T}_n), d_{\text{gr}}) &\xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{m}_\infty, D^*) \\ 3^{1/4} n^{-1/4} \cdot (V(\mathcal{T}_n), d_{\text{fpp}}) &\xrightarrow[n \rightarrow \infty]{(d)} \mathbf{c}_0 \cdot (\mathbf{m}_\infty, D^*) \\ 3^{1/4} n^{-1/4} \cdot (F(\mathcal{T}_n), d_{\text{gr}}^\dagger) &\xrightarrow[n \rightarrow \infty]{(d)} \mathbf{c}_1 \cdot (\mathbf{m}_\infty, D^*) \\ 3^{1/4} n^{-1/4} \cdot (F(\mathcal{T}_n), d_{\text{Eden}}^\dagger) &\xrightarrow[n \rightarrow \infty]{(d)} \mathbf{c}_2 \cdot (\mathbf{m}_\infty, D^*) \end{aligned}$$

hold jointly (with the same limit), in the the space of all isometry classes of compact metric spaces equipped with the Gromov–Hausdorff distance.

The first convergence in distribution of the corollary is proved in [27]. The other convergences, and the fact that they hold jointly with the first one, then follow from Theorems 1 and 3.

**Remark 8.** *As explained below in Appendix A1, the first convergence of the corollary holds in the stronger sense of the Gromov–Hausdorff–Prokhorov measure, if the sets  $V(\mathcal{T}_n)$  are equipped with the uniform probability measure and the Brownian map with its volume measure. It follows that the second convergence also holds in this stronger sense. The same should be true for the last two ones, but verifying this would require some additional work.*

We can also state an analog of Theorem 2. We let  $F(\mathcal{T}_\infty)$  be the set of all faces of the UIPT  $\mathcal{T}_\infty$ , and we equip this set with the dual graph distance  $d_{\text{gr}}^\dagger$  and with the Eden distance  $d_{\text{Eden}}^\dagger$  defined as above from independent exponential weights on the dual edges.

For every  $r \geq 0$ , we let  $B_r^{\text{dual}}(\mathcal{T}_\infty)$  (resp.  $B_r^{\text{Eden}}(\mathcal{T}_\infty)$ ) be the union of all faces of  $\mathcal{T}_\infty$  at dual graph distance (resp. at Eden distance) less than or equal to  $r$  from the root face (by definition, the root face is the face lying on the right of the root edge).

**Theorem 4.** *Let  $\varepsilon \in (0, 1)$ . With the same constants  $\mathbf{c}_1$  and  $\mathbf{c}_2$  as in Theorem 3 we have*

$$\lim_{r \rightarrow \infty} \mathbb{P} \left( \sup_{\substack{x, y \in V(B_r(\mathcal{T}_\infty)), f, g \in F(B_r(\mathcal{T}_\infty)) \\ x \triangleleft f, y \triangleleft g}} |d_{\text{gr}}^\dagger(f, g) - \mathbf{c}_1 d_{\text{gr}}(x, y)| > \varepsilon r \right) = 0,$$

$$\lim_{r \rightarrow \infty} \mathbb{P} \left( \sup_{\substack{x, y \in V(B_r(\mathcal{T}_\infty)), f, g \in F(B_r(\mathcal{T}_\infty)) \\ x \triangleleft f, y \triangleleft g}} |d_{\text{Eden}}^\dagger(f, g) - \mathbf{c}_2 d_{\text{gr}}(x, y)| > \varepsilon r \right) = 0.$$

Consequently,

$$\mathbb{P} \left( B_{(1-\varepsilon)r/\mathbf{c}_1}(\mathcal{T}_\infty) \subset B_r^{\text{dual}}(\mathcal{T}_\infty) \subset B_{(1+\varepsilon)r/\mathbf{c}_1}(\mathcal{T}_\infty) \right) \xrightarrow{r \rightarrow \infty} 1,$$

$$\mathbb{P} \left( B_{(1-\varepsilon)r/\mathbf{c}_2}(\mathcal{T}_\infty) \subset B_r^{\text{Eden}}(\mathcal{T}_\infty) \subset B_{(1+\varepsilon)r/\mathbf{c}_2}(\mathcal{T}_\infty) \right) \xrightarrow{r \rightarrow \infty} 1.$$

**Identification of the constants.** The proofs of Theorems 3 and 4 will be given in the next subsections, but we immediately explain how the values of the constants  $\mathbf{c}_1$  and  $\mathbf{c}_2$  can be derived by combining Theorem 4 with the results of [17].

**Theorem 5.** *The constants  $\mathbf{c}_1$  and  $\mathbf{c}_2$  of Theorems 3 and 4 are given by*

$$\mathbf{c}_1 = 1 + 2\sqrt{3} \quad \text{and} \quad \mathbf{c}_2 = 2\sqrt{3}.$$

*Proof.* We rely on results of [17] on asymptotics of the volume of hulls. For every  $r > 0$ , write  $B_r^{\bullet, \text{dual}}(\mathcal{T}_\infty)$  for the hull associated with  $B_r^{\text{dual}}(\mathcal{T}_\infty)$  (that is, the complement of the unbounded connected component of the complement of  $B_r^{\text{dual}}(\mathcal{T}_\infty)$ ), and similarly  $B_r^{\bullet, \text{Eden}}(\mathcal{T}_\infty)$  for the hull associated with  $B_r^{\text{Eden}}(\mathcal{T}_\infty)$ . Let  $|B_r^\bullet(\mathcal{T}_\infty)|$ , resp.  $|B_r^{\text{dual}}(\mathcal{T}_\infty)|$ , resp.  $|B_r^{\bullet, \text{Eden}}(\mathcal{T}_\infty)|$ , stand for the volume (number of faces) of  $B_r^\bullet(\mathcal{T}_\infty)$ , resp.  $B_r^{\bullet, \text{dual}}(\mathcal{T}_\infty)$ , resp.  $B_r^{\bullet, \text{Eden}}(\mathcal{T}_\infty)$ .

By [17, Theorems 2, 3 and 4], we have

$$\left( n^{-4} |B_{[nt]}^\bullet(\mathcal{T}_\infty)| \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left( \frac{64}{3} \mathcal{M}_t \right)_{t \geq 0} \quad (67)$$

$$\left( n^{-4} |B_{[nt]}^{\bullet, \text{dual}}(\mathcal{T}_\infty)| \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left( (1 + 2\sqrt{3})^{-4} \frac{64}{3} \mathcal{M}_t \right)_{t \geq 0}, \quad (68)$$

$$\left( n^{-4} |B_{[nt]}^{\bullet, \text{Eden}}(\mathcal{T}_\infty)| \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left( \frac{4}{27} \mathcal{M}_t \right)_{t \geq 0}. \quad (69)$$

where the process  $\mathcal{M}_t$ , which is defined in the introduction of [16, 17], satisfies the scaling property

$$(\mathcal{M}_{\lambda t})_{t \geq 0} \stackrel{(d)}{=} \lambda^4 (\mathcal{M}_t)_{t \geq 0},$$

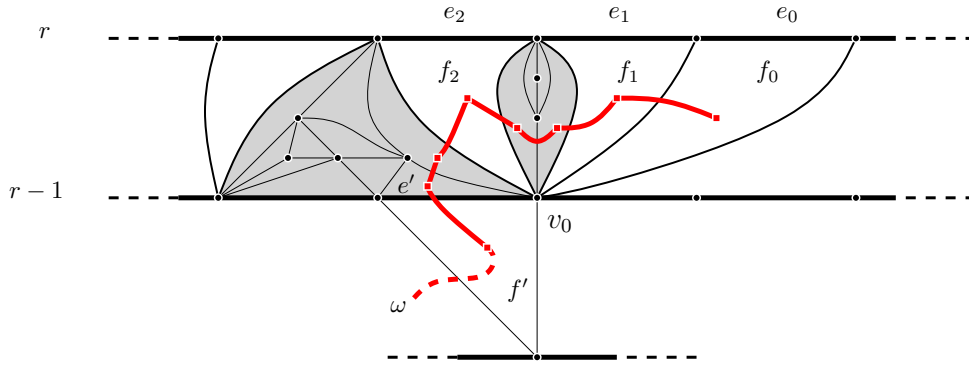
for every  $\lambda > 0$ . Note that Theorems 2, 3 and 4 in [17] give the preceding convergences in the case of the UIPT of type II, but Section 6.1 in [17] explains how these statements can be extended to the UIPT of type I, and gives the values of the constants arising in this case.

On the other hand, (67) and Theorem 4 imply that the limit in distribution (at least in the sense of finite-dimensional marginals) of  $(n^{-4}|B_{[nt]}^{\bullet, \text{Eden}}(\mathcal{T}_\infty)|)_{t \geq 0}$  is the process  $(\frac{64}{3}\mathcal{M}_{t/c_2})_{t \geq 0}$ , which has the same distribution as  $(\frac{64}{3}(\mathbf{c}_2)^{-4}\mathcal{M}_t)_{t \geq 0}$ . Comparing with (69), we get that  $\frac{64}{3}(\mathbf{c}_2)^{-4} = \frac{4}{27}$ , hence  $\mathbf{c}_2 = 2\sqrt{3}$ . Similarly, the value  $\mathbf{c}_1 = 1 + 2\sqrt{3}$  is derived by comparing (67) and (68).  $\square$

In the remaining part of this section, we explain the proof of Theorems 3 and 4. The general outline is the same as for Theorems 1 and 2, but some additional ingredients are needed.

## 7.2 Preliminary estimates

As in the previous sections, we discuss the UIPT before considering finite triangulations. In order to get upper bounds on the distances  $d_{\text{gr}}^\dagger$  and  $d_{\text{Eden}}^\dagger$ , it will be convenient to consider certain special paths in  $\mathcal{F}(\mathcal{T}_\infty^{(1)})$ . For every  $r \geq 1$ , we let  $\mathcal{F}_r(\mathcal{T}_\infty^{(1)})$  be the set of all downward triangles at height  $r$  in  $\mathcal{T}_\infty^{(1)}$ . A *downward path* is a dual path  $\omega$  that starts from some  $f_0 \in \mathcal{F}_r(\mathcal{T}_\infty^{(1)})$  and ends at the bottom face, which is constructed in the following way, see Fig. 11. Let  $v_0$  be the unique vertex of  $\partial^* B_{r-1}^\bullet(\mathcal{T}_\infty^{(1)})$  that is incident to  $f_0$ , let  $e_0$  be the edge of  $\partial^* B_r^\bullet(\mathcal{T}_\infty^{(1)})$  incident to  $f_0$  and let  $e_0, e_1, \dots$  be the sequence of edges of  $\partial^* B_r^\bullet(\mathcal{T}_\infty^{(1)})$  listed in counterclockwise order from  $e_0$  (recall our orientation of the cycles  $\partial^* B_j^\bullet(\mathcal{T}_\infty^{(1)})$  in clockwise order). Let  $e_N$  ( $N \geq 0$ ) be the first one in this list that has at least one child in the skeleton of  $B_r^\bullet(\mathcal{T}_\infty^{(1)})$ , and let  $e'$  be the unique edge of  $\partial^* B_{r-1}^\bullet(\mathcal{T}_\infty^{(1)})$  whose terminal vertex is  $v_0$ . Let  $f_0, f_1, \dots, f_N$  and  $f'$  be the downward triangles associated respectively with  $e_0, \dots, e_N$  and  $e'$ . Notice that  $v_0$  is incident to all the faces  $f_0, f_1, \dots, f_N$  and  $f'$ . Our dual path  $\omega$  will visit successively the faces  $f_0, f_1, \dots, f_N$  and  $f'$ . Between the visits of  $f_i$  and  $f_{i+1}$ , for  $0 \leq i \leq N-1$ , or the visits of  $f_N$  and  $f'$ , the path “crosses” the slot of boundary size 2 (or size  $c_{e_N} + 2$  for the last one) between these two faces: It does so by turning in counterclockwise order around  $v_0$ , visiting successively all faces of the triangulation filling in the slot that are incident to  $v_0$  (see Fig. 11, where  $N = 2$ ). We have just described how the downward path  $\omega$  goes from  $f_0$  to a certain triangle  $f' \in \mathcal{F}_{r-1}(\mathcal{T}_\infty^{(1)})$ , but we can now continue the construction by induction until we reach the bottom face. Notice that this downward path is in general not a geodesic for the dual metric.



**Figure 11:** Illustration of the construction of the downward path (in red).

Similarly, we can define downward paths in the lower half-plane model  $\mathcal{L}$ . For every downward triangle  $f$  incident to  $\mathcal{L}_0$ , there exists such a dual path connecting  $f$  to a certain downward triangle  $f'$  incident to  $\mathcal{L}_r$ , for some  $r \geq 1$ . This path is constructed in exactly the same way as explained above for the UIPT.

Notice that the time needed by a downward path to cross a slot is exactly equal to the degree of the root vertex of the triangulation filling in the slot (see subsection 2.2 for the definition of this root vertex).

Tail estimates for the latter quantity are given in Proposition 30 in Appendix A2 below, and are used in the next lemma to bound the length of downward paths, first in the easier case of the LHPT. For every  $i \in \mathbb{Z}$  and every  $r \geq 0$ , we write  $f_{(i,r)}$  for the unique downward triangle of  $\mathcal{L}$  that is incident to the edge between  $(i-1, -r)$  and  $(i, -r)$ .

**Lemma 24.** *Let  $\omega_r$  be the downward path in  $\mathcal{L}$  connecting  $f_{(0,0)}$  to a downward triangle incident to  $\mathcal{L}_r$ , and write  $|\omega_r|$  for the length of  $\omega_r$ . There exist two constants  $\mu > 0$  and  $K < \infty$  such that for every integer  $r \geq 1$ ,*

$$\mathbb{E}[\exp(\mu|\omega_r|)] \leq K^r.$$

*Proof.* By the independence properties of the LHPT, it is enough to consider the case  $r = 1$  and to prove that  $\mathbb{E}[\exp(\mu|\omega_1|)] < \infty$  for some  $\mu > 0$ . Note that the path  $\omega_1$  connects  $f_{(0,0)}$  to  $f_{(0,1)}$ . By the definition of downward paths,  $\omega_1$  visits successively  $f_{(0,0)}, f_{(-1,0)}, \dots, f_{(-N,0)}$  and  $f_{(0,1)}$ , where the triangles  $f_{(0,0)}, f_{(-1,0)}, \dots, f_{(-N,0)}$  are incident to  $(0, -1)$ , but  $f_{(-N-1,0)}$  is not (in the construction of  $\mathcal{L}$ ,  $N+1$  is the first positive integer  $i$  such that the tree  $\mathcal{T}_{-i}$  is non trivial). Furthermore, the construction of  $\mathcal{L}$  shows that, for every  $k \geq 0$ ,

$$\mathbb{P}(N \geq k) = \theta(0)^k.$$

Conditionally on the forest  $(\mathcal{T}_i)_{i \in \mathbb{Z}}$  (and in particular on  $N$ ), the slots associated with the downward triangles  $f_{(0,0)}, f_{(-1,0)}, \dots, f_{(-N+1,0)}$  are filled in by independent Boltzmann triangulations of the 2-gon, and the slot associated with  $f_{(-N,0)}$  is filled in by an independent Boltzmann triangulation of the  $d+2$ -gon, where  $d$  is the number of children of the root of  $\mathcal{T}_{-N-1}$ . It then follows that we have

$$|\omega_1| = \mathcal{D}_2^{(0)} + \dots + \mathcal{D}_2^{(N-1)} + \mathcal{D}_{d+2},$$

where, conditionally on  $(\mathcal{T}_i)_{i \in \mathbb{Z}}$ , the variables  $\mathcal{D}_2^{(0)}, \dots, \mathcal{D}_2^{(N-1)}$  and  $\mathcal{D}_{d+2}$  are independent,  $\mathcal{D}_2^{(0)}, \dots, \mathcal{D}_2^{(N-1)}$  are distributed as the degree of the root vertex in a Boltzmann triangulation of the 2-gon, and  $\mathcal{D}_{d+2}$  is distributed as the degree of the root vertex in a Boltzmann triangulation of the  $d+2$ -gon. By Proposition 30 we can choose  $\beta > 0$  small enough and a finite constant  $C$  (not depending on  $d$ ) such that  $\mathbb{E}[\exp(\beta \mathcal{D}_2^{(0)})] \leq C$  and  $\mathbb{E}[\exp(\beta \mathcal{D}_{d+2})] \leq C$ . Then, if  $0 < \mu \leq \beta$ , we have

$$\mathbb{E}[\exp(\mu|\omega_1|)] \leq C(1 - \theta(0)) \sum_{k=0}^{\infty} \theta(0)^k \mathbb{E}[\exp(\mu \mathcal{D}_2^{(0)})]^k$$

and we get the desired result by choosing  $\mu > 0$  small enough so that  $\theta(0) \mathbb{E}[\exp(\mu \mathcal{D}_2^{(0)})] < 1$ .  $\square$

We can now state an analog of Proposition 18. For every  $r \geq 0$ , we write  $\mathcal{L}_r^\dagger$  for the collection of all downward triangles incident to an edge of  $\mathcal{L}_r$  in the lower half-plane model. We assume that the dual graph of  $\mathcal{L}$  is equipped with the graph distance  $d_{\text{gr}}^\dagger$  and the Eden distance  $d_{\text{Eden}}^\dagger$  defined as previously.

**Proposition 25.** *There exist two constants  $\mathbf{c}_1 \geq 2$  and  $\mathbf{c}_2 > 0$  such that*

$$\begin{aligned} r^{-1} d_{\text{gr}}^\dagger(f_{(0,0)}, \mathcal{L}_r^\dagger) &\xrightarrow[r \rightarrow \infty]{\text{a.s.}} \mathbf{c}_1, \\ r^{-1} d_{\text{Eden}}^\dagger(f_{(0,0)}, \mathcal{L}_r^\dagger) &\xrightarrow[r \rightarrow \infty]{\text{a.s.}} \mathbf{c}_2. \end{aligned}$$

The proof is very similar to that of Proposition 18 and the details are left to the reader. In order to apply the subadditive ergodic theorem, we need the fact that  $\mathbb{E}[d_{\text{gr}}^\dagger(f_{(0,0)}, \mathcal{L}_r^\dagger)] < \infty$ , which follows from Lemma 24. The property  $\mathbf{c}_1 \geq 2$  is obvious since we have  $d_{\text{gr}}^\dagger(f_{(0,0)}, \mathcal{L}_r^\dagger) \geq 2r$ . The fact that  $\mathbf{c}_2$  is (strictly) positive is not completely obvious but can be verified as follows. Since we are dealing with

triangulations, there are at most  $3^{2r}$  distinct injective dual paths of length  $2r$  starting from  $f_{(0,0)}$  in  $\mathcal{L}$ . However, a crude large deviation argument shows that, if  $\delta > 0$  is small enough, a.s. for all sufficiently large  $r$ , none of these injective dual paths can have a total Eden weight smaller than  $\delta r$ . It follows that  $\mathbf{c}_2 \geq \delta > 0$ .

### 7.3 Technical lemmas

It will be important to have a good control of the dual distances  $d_{\text{gr}}^\dagger$  and  $d_{\text{Eden}}^\dagger$  in terms of the graph distance on the original graph. This is the goal of the two technical lemmas of this section. The first one deals with the case of the UIPT and the second one with finite triangulations.

For every integer  $r \geq 1$ , we let  $f_r$  be a downward triangle at height  $r$  chosen uniformly at random in  $\mathbb{F}_r(\mathcal{T}_\infty^{(1)})$ .

**Lemma 26.** *In the UIPT of the 1-gon  $\mathcal{T}_\infty^{(1)}$ , there exist positive constants  $K, \alpha, \beta$  such that, for every integers  $0 \leq r < s$ ,*

$$\begin{aligned} \mathbb{P}\left(d_{\text{gr}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s-r)\right) &\leq K e^{-\beta(s-r)}, \\ \mathbb{P}\left(d_{\text{Eden}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s-r)\right) &\leq K e^{-\beta(s-r)}, \end{aligned}$$

where  $B_0^\bullet(\mathcal{T}_\infty^{(1)})$  should be interpreted as the bottom face.

*Proof.* It is enough to treat the case of  $d_{\text{gr}}^\dagger$ . Indeed, by considering the weights along a geodesic dual path from  $f_s$  to  $B_r^\bullet(\mathcal{T}_\infty^{(1)})$ , one immediately gets that, for every  $\alpha' > \alpha$ ,

$$\mathbb{P}\left(d_{\text{gr}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s-r), d_{\text{Eden}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha'(s-r)\right)$$

is bounded above, for  $s-r$  sufficiently large, by  $\exp(-\beta'(s-r))$  for some constant  $\beta' > 0$ .

So we deal with  $d_{\text{gr}}^\dagger$ , and we assume that  $r \geq 1$  (the case  $r = 0$  is exactly similar). Recall the notation introduced before Proposition 5:  $\tilde{\mathcal{F}}_{r,s}^{(1)}$  is the skeleton of  $B_s^\bullet(\mathcal{T}_\infty^{(1)}) \setminus B_r^\bullet(\mathcal{T}_\infty^{(1)})$  reordered via a random cyclic permutation, and where the distinguished vertex at height  $s-r$  has been “forgotten”. We may assume that  $f_s$  is the downward triangle corresponding to the root of the last tree in  $\tilde{\mathcal{F}}_{r,s}^{(1)}$ . In the proof of Proposition 5, we observed that, for every  $p, q \geq 1$ , we have

$$\mathbb{P}\left(\tilde{\mathcal{F}}_{r,s}^{(1)} = \mathcal{F} \mid L_r = p\right) = \frac{h(q)}{h(p)} \prod_{v \in \mathcal{F}^*} \theta(c_v),$$

for every fixed forest  $\mathcal{F} \in \mathbb{F}_{p,q,s-r}''$ . Furthermore, the law of  $L_r$  was obtained in the proof of Lemma 4:

$$\mathbb{P}(L_r = p) = \frac{h(p)}{h(1)} \mathbb{P}_p(Y_r = 1) = \frac{h(p)}{h(1)} \frac{p}{(r+1)^3} \left(1 - (r+1)^{-2}\right)^{p-1}. \quad (70)$$

It follows that, for every forest  $\mathcal{F} \in \mathbb{F}_{p,q,s-r}''$

$$\mathbb{P}\left(\tilde{\mathcal{F}}_{r,s}^{(1)} = \mathcal{F}\right) = \frac{h(q)}{h(1)} \frac{p}{(r+1)^3} \left(1 - (r+1)^{-2}\right)^p \prod_{v \in \mathcal{F}^*} \theta(c_v) \leq \frac{C}{\sqrt{q}(r+1)} \prod_{v \in \mathcal{F}^*} \theta(c_v), \quad (71)$$

for some finite constant  $C$  (we use the fact that

$$\frac{p}{(r+1)^2} \left(1 - (r+1)^{-2}\right)^p \leq \frac{p}{(r+1)^2} \exp\left(-\frac{p}{(r+1)^2}\right)$$

is bounded above by a constant).

It follows from (71) that the law of  $\tilde{\mathcal{F}}_{r,s}^{(1)}$  under  $\mathbb{P}(\cdot \cap \{L_s = q\})$  is dominated by  $C/(\sqrt{q}(r+1))$  times the law of a forest of  $q$  independent Galton–Watson trees with offspring distribution  $\theta$  truncated at height  $s-r$ , and we may restrict the latter law to the event where the truncated forest has height exactly  $s-r$ . Now note that the length of the downward path from  $f_s$  to  $B_r^\bullet(\mathcal{T}_\infty^{(1)})$  is determined by the forest  $\tilde{\mathcal{F}}_{r,s}^{(1)}$  and by the triangulations with a boundary filling in the slots associated with the vertices of this forest at height strictly less than  $s-r$ . It follows that the law of this length under  $\mathbb{P}(\cdot \cap \{L_s = q\})$  is dominated by  $C/(\sqrt{q}(r+1))$  times the law of the length of the downward path that one would get by considering a triangulation of the cylinder of height  $s-r$  whose (cyclically permuted) skeleton is a forest of  $q$  independent Galton–Watson trees with offspring distribution  $\theta$  truncated at height  $s-r$  (and we restrict our attention to the event where the truncated forest has height  $s-r$ ), and whose slots are filled in by independent Boltzmann triangulations. In the latter model, the numbers of downward triangles that the downward path crosses in each layer are independent variables, and so are the sizes of the slots which are not 2-gons crossed in the different layers. These considerations show that the law under  $\mathbb{P}(\cdot \cap \{L_s = q\})$  of the length of the downward path from  $f_s$  to  $B_r^\bullet(\mathcal{T}_\infty^{(1)})$  is dominated by  $C/(\sqrt{q}(r+1))$  times the law of the downward path from  $f_{(0,0)}$  to  $\mathcal{L}_{s-r}$  in the LHPT model. Using Lemma 24, we now get

$$\begin{aligned} \mathbb{P}(\{d_{\text{gr}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s-r)\} \cap \{L_s = q\}) &\leq \frac{C}{\sqrt{q}(r+1)} \mathbb{P}(|\omega_{s-r}| > \alpha(s-r)) \\ &\leq \frac{C}{\sqrt{q}(r+1)} e^{-\mu\alpha(s-r)} K^{s-r}. \end{aligned}$$

We can now fix  $\alpha > 0$  and  $\beta' > 0$  such that  $e^{-\mu\alpha}K \leq e^{-\beta'}$ , and we get

$$\mathbb{P}(\{d_{\text{gr}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s-r)\} \cap \{L_s = q\}) \leq \frac{C}{\sqrt{q}(r+1)} \exp(-\beta'(s-r)). \quad (72)$$

To complete the argument we need to sum over the possible values of  $q$ . Using (72) for  $q \leq (s-r)s^2$  and (23) for  $q > (s-r)s^2$  we get that

$$\begin{aligned} \mathbb{P}(\{d_{\text{gr}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s-r)\}) &\leq \mathbb{P}(L_s > (s-r)s^2) + \sum_{q=1}^{(s-r)s^2} \frac{C}{\sqrt{q}(r+1)} \exp(-\beta'(s-r)) \\ &\leq C_0 \exp(-(s-r)/5) + 2C \frac{\sqrt{(s-r)s^2}}{r+1} \exp(-\beta'(s-r)) \\ &\leq C_0 \exp(-(s-r)/5) + 4C(s-r)^{3/2} \exp(-\beta'(s-r)). \end{aligned}$$

Taking  $\beta \in (0, \frac{1}{5} \wedge \beta')$ , the last display is bounded above by  $C' \exp(-\beta(s-r))$  for some constant  $C' > 0$ .  $\square$

**Remark 9.** As the proof shows, we can replace  $d_{\text{gr}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)}))$  (resp.  $d_{\text{Eden}}^\dagger(f_s, B_r^\bullet(\mathcal{T}_\infty^{(1)}))$ ) in the statement of Lemma 26 by the length (resp. the weight) of the downward path from  $f_s$  to  $B_r^\bullet(\mathcal{T}_\infty^{(1)})$ . The same remark holds for the next corollary.

**Corollary 27.** Let  $\alpha$  be as in the preceding lemma, and let  $\delta > 0$ . For every integer  $R \geq 1$ , let  $A_R(\delta)$  be the event where the property

$$d_{\text{gr}}^\dagger(f, B_r^\bullet(\mathcal{T}_\infty^{(1)})) \leq \alpha(s-r)$$

holds for every  $0 \leq r < s \leq R$  such that  $s-r \geq \delta R$ , for every downward triangle  $f$  at height  $s$ . There exists a constant  $\tilde{\beta} > 0$  such that, for every sufficiently large  $R$ ,

$$\mathbb{P}(A_R(\delta)) \geq 1 - e^{-\tilde{\beta}R}.$$

The same result holds if  $d_{\text{dual}}^\dagger$  is replaced by  $d_{\text{Eden}}^\dagger$ .



*Proof.* Consider first fixed values of  $r$  and  $s$  such that  $0 \leq r < s \leq R$  and  $s - r \geq \delta R$ . Let  $f_{(1)}$  be uniformly distributed over  $F_s(\mathcal{T}_\infty^{(1)})$ , and define  $f_{(1)}, f_{(2)}, f_{(3)}, \dots$  as the successive downward triangles at height  $s$  visited when moving around  $\partial B_s^\bullet(\mathcal{T}_\infty^{(1)})$  in clockwise order, starting from  $f_{(1)}$ . For every integer  $j \geq 1$ ,  $f_{(j)}$  is also uniformly distributed over  $F_s(\mathcal{T}_\infty^{(1)})$ . By Lemma 26,

$$\mathbb{P}\left(d_{\text{gr}}^\dagger(f_{(j)}, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s - r)\right) \leq K e^{-\beta \delta R}.$$

Using the bound (23), it follows that

$$\begin{aligned} & \mathbb{P}\left(d_{\text{gr}}^\dagger(f, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s - r), \text{ for some downward triangle } f \text{ at height } s\right) \\ & \leq \sum_{j=1}^{Rs^2} \mathbb{P}\left(d_{\text{gr}}^\dagger(f_{(j)}, B_r^\bullet(\mathcal{T}_\infty^{(1)})) > \alpha(s - r)\right) + \mathbb{P}(L_s > Rs^2) \\ & \leq K R^3 \exp(-\beta \delta R) + C_0 \exp(-R/5). \end{aligned}$$

To get the estimate of the corollary, we only need to sum this bound over the possible values of  $r$  and  $s$ . The proof for  $d_{\text{Eden}}^\dagger$  is exactly the same.  $\square$

We now turn to finite triangulations. As previously,  $\mathcal{T}_n$  denotes a uniform rooted plane triangulation with  $n + 1$  vertices. The next lemma gives a uniform estimate for the dual (or Eden) distance on  $F(\mathcal{T}_n)$  in terms of the graph distance on  $V(\mathcal{T}_n)$ . Recall our notation  $x \triangleleft f$  meaning that the vertex  $x$  is incident to the face  $f$ .

**Lemma 28.** *Let  $\alpha$  be as in Lemma 26. Let  $\varepsilon \in (0, 1/4)$ , and for every integer  $n \geq 1$ , let  $E_n$  be the event where the bound*

$$d_{\text{gr}}^\dagger(f, g) \leq \alpha d_{\text{gr}}(x, y) + n^\varepsilon$$

*holds for every  $x, y \in V(\mathcal{T}_n)$  and  $f, g \in F(\mathcal{T}_n)$  such that  $x \triangleleft f$  and  $y \triangleleft g$ . Then*

$$\mathbb{P}(E_n) \xrightarrow{n \rightarrow \infty} 1.$$

*The same result holds if  $d_{\text{gr}}^\dagger$  is replaced by  $d_{\text{Eden}}^\dagger$ .*

*Proof.* Recall the notation of Proposition 21:  $\overline{\mathcal{T}}_n^{(1)}$  is a uniform pointed triangulation of the 1-gon with  $n$  inner vertices, whose root vertex is  $\rho_n$  and the distinguished vertex is denoted by  $o_n$ . To simplify notation, set  $d_n = d_{\text{gr}}(\rho_n, o_n)$ . We can make sense of the hull  $B_r^\bullet(\overline{\mathcal{T}}_n^{(1)})$  provided that  $0 < r < d_n$ . Then write  $\Theta$  for the set of all triangulations  $\mathbf{t}$  that belong to  $\mathbb{C}_{1,r}$  for some  $r > 1$  and are such that there exists a face  $f$  incident to  $\partial^* \mathbf{t}$  whose dual graph distance from the bottom face is strictly greater than  $\alpha r$ .

Using Lemma 22, we have

$$\begin{aligned} \mathbb{P}\left(d_n > r; B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) \in \Theta\right) & \leq \sum_{\mathbf{t} \in \mathbb{C}_{1,r}} \mathbf{1}_\Theta(\mathbf{t}) \mathbb{P}(d_n > r; B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) = \mathbf{t}) \\ & \leq \bar{c} n^{3/2} \sum_{\mathbf{t} \in \mathbb{C}_{1,r}} \mathbf{1}_\Theta(\mathbf{t}) \mathbb{P}(B_r^\bullet(\mathcal{T}_\infty^{(1)}) = \mathbf{t}) \\ & \leq \bar{c} n^{3/2} \exp(-\tilde{\beta} r), \end{aligned}$$

where the last inequality follows from Corollary 27. We can sum this bound from  $r = \lfloor n^\varepsilon \rfloor$  to  $\infty$ , to get

$$\mathbb{E} \left[ \sum_{r=\lfloor n^\varepsilon \rfloor}^{\infty} \mathbf{1}_{\{r < d_n; B_r^\bullet(\overline{\mathcal{T}}_n^{(1)}) \in \Theta\}} \right] \leq \tilde{c} \exp(-a n^\varepsilon),$$

with some other constants  $\tilde{c} > 0, a > 0$ . It follows that

$$\mathbb{P}\left(d_n > n^\varepsilon; B_{d_n-1}^\bullet(\overline{\mathcal{T}}_n^{(1)}) \in \Theta\right) \leq \tilde{c} \exp(-a n^\varepsilon). \quad (73)$$

Then notice that  $o_n$  is adjacent to a vertex  $v_0$  that belongs to the boundary of  $B_{d_n-1}^\bullet(\overline{\mathcal{T}}_n^{(1)})$  (take  $v_0$  on a geodesic from  $o_n$  to  $\rho_n$ ). If  $g$  is any face incident to  $o_n$  and if  $g'$  is a face incident to an edge between  $o_n$  and  $v_0$ , the dual graph distance between  $g$  and  $g'$  is bounded above by the degree of  $o_n$  and thus by the maximal vertex degree of  $\mathcal{T}_n^{(1)}$ , which we denote by  $\text{MD}(\mathcal{T}_n^{(1)})$ . For the same reason, the dual graph distance between  $g'$  and a downward triangle at height  $d_n - 1$  incident to  $v_0$  is bounded by  $\text{MD}(\mathcal{T}_n^{(1)})$ , and so is the dual graph distance between any face  $f$  incident to  $\rho_n$  and the bottom face. The preceding considerations and the definition of  $\Theta$  show that, on the event  $\{d_n > n^\varepsilon; B_{d_n-1}^\bullet(\overline{\mathcal{T}}_n^{(1)}) \notin \Theta\}$ , we have  $d_{\text{gr}}^\dagger(f, g) \leq \alpha d_n + 3\text{MD}(\mathcal{T}_n^{(1)})$ , whenever  $\rho_n \triangleleft f$  and  $o_n \triangleleft g$ .

Obviously, the same result holds if one considers instead the rooted and pointed plane triangulation  $\overline{\mathcal{T}}_n$  constructed from  $\overline{\mathcal{T}}_n^{(1)}$  via the transformation of Fig. 2. Now re-root  $\overline{\mathcal{T}}_n$  at an oriented edge  $e_n$  chosen uniformly and independently of  $o_n$ , and write  $\overline{\mathcal{T}}'_n$  for the resulting rooted and pointed planar map. Then,  $\overline{\mathcal{T}}'_n$  has the same distribution as  $\overline{\mathcal{T}}_n$  and writing  $\rho'_n$  for the initial vertex of  $e_n$ , and  $d'_n = d_{\text{gr}}(\rho'_n, o_n)$ , we have from (73)

$$\mathbb{P}\left(d'_n > n^\varepsilon; d_{\text{gr}}^\dagger(f, g) > \alpha d'_n + 3\text{MD}(\overline{\mathcal{T}}_n) \text{ whenever } \rho'_n \triangleleft f \text{ and } o_n \triangleleft g\right) \leq \tilde{c} \exp(-a n^\varepsilon).$$

By the same considerations as in the beginning of the proof of Theorem 1, this implies

$$\mathbb{E}\left[\sum_{v, v' \in V(\mathcal{T}_n)} \mathbf{1}_{\{d_{\text{gr}}(v, v') > n^\varepsilon\}} \mathbf{1}_{\{d_{\text{gr}}^\dagger(f, g) > \alpha d_{\text{gr}}(v, v') + 3\text{MD}(\mathcal{T}_n) \text{ whenever } v \triangleleft f \text{ and } v' \triangleleft g\}}\right] \leq 6(n+1)^2 \tilde{c} \exp(-a n^\varepsilon).$$

Hence, with probability tending to 1 as  $n \rightarrow \infty$ , we have the bound

$$d_{\text{gr}}^\dagger(f, g) > \alpha d_{\text{gr}}(v, v') + 3\text{MD}(\mathcal{T}_n)$$

whenever  $v, v' \in V(\mathcal{T}_n)$ ,  $d_{\text{gr}}(v, v') > n^\varepsilon$  and  $v \triangleleft f$ ,  $v' \triangleleft g$ . However, Lemma 32 in Appendix A2 below shows that we can find a constant  $A > 0$  such that the bound  $\text{MD}(\mathcal{T}_n) \leq A \log n$  holds with probability tending to 1 as  $n \rightarrow \infty$ . Combining this bound with the previous display, we get the desired result, except for the restriction  $d_{\text{gr}}(v, v') > n^\varepsilon$ . However, if  $d_{\text{gr}}(v, v') \leq n^\varepsilon$ , we can just use the simple bound  $d_{\text{gr}}^\dagger(f, g) \leq \text{MD}(\mathcal{T}_n)(d_{\text{gr}}(v, v') + 1)$ . This completes the proof of the result for  $d_{\text{gr}}^\dagger$ . The case of  $d_{\text{Eden}}^\dagger$  is treated in a similar way, using also the fact that the maximal weight of a dual edge in  $\mathcal{T}_n$  can be bounded by  $n^\varepsilon$  outside a set of probability tending to 0 as  $n \rightarrow \infty$ .  $\square$

## 7.4 Proof of the theorems about dual distances

Lemma 26 and Lemma 28 provide the technical ingredients that are needed to extend the arguments of the proofs of Theorems 1 and 2 to the setting of Theorems 3 and 4. In the present subsection, we briefly explain the necessary adaptations of the proofs.

Recall the constants  $\mathbf{c}_1$  and  $\mathbf{c}_2$  from Proposition 25. Let us state an analog of Propositions 19 and 20. Recall that we interpret  $B_0^\bullet(\mathcal{T}_\infty^{(1)})$  as the bottom face.

**Proposition 29.** *Let  $\varepsilon > 0$  and  $\delta > 0$ . We can find  $\eta \in (0, \frac{1}{2})$  such that, for every sufficiently large  $n$ , the bounds*

$$(1 - \varepsilon)\mathbf{c}_1 \lfloor \eta n \rfloor \leq d_{\text{gr}}^\dagger(f, B_{n - \lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})) \leq (1 + \varepsilon)\mathbf{c}_1 \lfloor \eta n \rfloor$$

hold for every  $f \in F_n(\mathcal{T}_\infty^{(1)})$ , with probability at least  $1 - \delta$ . Furthermore,

$$\mathbb{P}\left((\mathbf{c}_1 - \varepsilon)n \leq d_{\text{gr}}^\dagger(B_0^\bullet(\mathcal{T}_\infty^{(1)}), f) \leq (\mathbf{c}_1 + \varepsilon)n, \text{ for every } f \in F_n(\mathcal{T}_\infty^{(1)})\right) \xrightarrow{n \rightarrow \infty} 1.$$

The same properties hold when  $d_{\text{gr}}^\dagger$  is replaced by  $d_{\text{Eden}}^\dagger$  provided that  $\mathbf{c}_1$  is replaced by  $\mathbf{c}_2$ .

*Proof (sketch).* We start with the first assertion. Fix  $n$  and first choose  $f$  uniformly at random in  $F_n(\mathcal{T}_\infty^{(1)})$ . We argue in a very similar way as in the proof of Proposition 19, using Proposition 25 instead of Proposition 18, and noting that Corollary 27 already gives us the bound  $d_{\text{gr}}^\dagger(f, B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})) \leq \alpha \lfloor \eta n \rfloor$  outside a set of small probability. Recall the notation of the proof of Proposition 19, and, for every  $i \in \mathbb{Z}$ , write  $f_i^{(n)} \in F_n(\mathcal{T}_\infty^{(1)})$  for the unique downward triangle incident to the edge of  $\partial B_n^\bullet(\mathcal{T}_\infty^{(1)})$  from  $u_i^{(n)}$  to  $u_{i+1}^{(n)}$ . Let  $j$  be such that  $f = f_j^{(n)}$ . We need to bound the probability that, for some  $i$  with  $j - an^2/16 \leq i \leq j + an^2/16$ , there is a dual path from  $f_i^{(n)}$  to  $B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})$  with length smaller than  $4\alpha\eta n$ , which stays in  $B_n^\bullet(\mathcal{T}_\infty^{(1)})$  and exits the region  $\mathcal{G}_j^{(n)}(\eta)$  before hitting  $B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})$ . However, a simple argument shows that, if there exists such a dual path, there will also exist a path (in the primal graph) from  $u_i^{(n)}$  to  $\partial_\ell \mathcal{G}_j^{(n)}(\eta)$  in  $B_n^\bullet(\mathcal{T}_\infty^{(1)}) \setminus B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})$ , with length smaller than  $4\alpha\eta n + 1$ , and we know from the proof of Proposition 19 that this cannot occur except on a set of small probability. To get a similar estimate in the case of  $d_{\text{Eden}}^\dagger$ , we need an additional ingredient. Precisely, the same large deviation argument as in the proof of Proposition 25 allows us to verify the existence of a constant  $\gamma > 0$  such that, except on a set of probability tending to 0 as  $k \rightarrow \infty$ , any injective dual path of length  $k$  starting from  $F_n(\mathcal{T}_\infty^{(1)})$  will have total Eden weight at least  $\gamma k$  (the point is that there are less than  $3^k$  such paths with a given starting face). So, except on a set of probability tending to 0 as  $n \rightarrow \infty$ , the existence of a dual path (which can be assumed to be injective) from  $f_i^{(n)}$  to  $B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})$  with Eden weight smaller than  $4(\alpha/\gamma)\eta n$ , which stays in  $B_n^\bullet(\mathcal{T}_\infty^{(1)})$  and exits the region  $\mathcal{G}_j^{(n)}(\eta)$  before hitting  $B_{n-\lfloor \eta n \rfloor}^\bullet(\mathcal{T}_\infty^{(1)})$ , implies that the same dual path has length smaller than  $4\alpha\eta n$ , and we can use the first part of the argument.

When adapting the final part of the proof of Proposition 19, we also need to verify that, for every  $\beta > 0$ , we can find an integer  $A$  sufficiently large so that, except on a set of probability tending to 0 as  $n \rightarrow \infty$ , any downward triangle at height  $n$  is connected to one of the downward triangles  $f_j^{(n)}$ ,  $0 \leq j \leq \lfloor a^{-1}A \rfloor$ , by a dual path in  $B_n^\bullet(\mathcal{T}_\infty^{(1)})$  with length (or Eden weight) at most  $\beta \mathbf{c}_1 \eta n$  ( $\beta \mathbf{c}_2 \eta n$  in the case of the Eden weight). To this end, we use again Proposition 17. We observe that if  $f = f_j^{(n)}$  and  $f' = f_{j'}^{(n)}$  are two downward triangles at height  $n$ , the fact that the left-most geodesics from  $u_j^{(n)}$  and  $u_{j'}^{(n)}$  coalesce above height  $n' < n$  implies that the same property holds for the downward paths from  $f$  and from  $f'$ . We can then use the bounds on the lengths of downward paths obtained in subsection 7.3 (see the remark preceding Corollary 27) to get the desired control on the length (or Eden weight) of the dual path from  $f$  to  $f'$  obtained by the concatenation of the respective downward paths from  $f$  and  $f'$  up to their coalescence time.

The proof of the second assertion of the proposition is similar to that of Proposition 20. We now use Corollary 27 to handle the “bad” values of  $i$  for which the bound

$$(1 - \varepsilon)\mathbf{c}_1(n_i - n_{i+1}) \leq d_{\text{gr}}^\dagger(f, B_{n_{i+1}}^\bullet(\mathcal{T}_\infty^{(1)})) \leq (1 + \varepsilon)\mathbf{c}_1(n_i - n_{i+1}), \text{ for every } f \in F_{n_i}(\mathcal{T}_\infty^{(1)})$$

fails (with the notation of the proof of Proposition 20). The remaining part of the argument is the same.  $\square$

*Proof of Theorem 3.* We start by deriving an analog of Proposition 21. Let  $f_*$  stands for the bottom face of  $\mathcal{T}_n^{(1)}$ . Let  $o_n$  be distributed uniformly on  $V(\mathcal{T}_n^{(1)})$  and let  $f_n$  be a face incident to  $o_n$  (which may be

fixed in some deterministic manner given  $o_n$ ). Then

$$\mathbb{P}\left(|d_{\text{gr}}^\dagger(f_*, f_n) - \mathbf{c}_1 d_{\text{gr}}(\rho_n, o_n)| > \varepsilon n^{1/4}\right) \xrightarrow{n \rightarrow \infty} 0, \quad (74)$$

and similarly if  $d_{\text{gr}}^\dagger$  is replaced by  $d_{\text{Eden}}^\dagger$  provided  $\mathbf{c}_1$  is replaced by  $\mathbf{c}_2$ . The proof is essentially the same as that of Proposition 21, using the same absolute continuity argument (justified by Lemma 22) but relying now on the second assertion of Proposition 29 instead of Proposition 20. The only notable modification is at the end of the proof where, on the event  $\{\beta_j n^{1/4} < d_{\text{gr}}(\rho_n, o_n) \leq \gamma_j n^{1/4}\}$ , we now use Lemma 28 to get an upper bound on  $d_{\text{gr}}^\dagger(f_n, B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\mathcal{T}_n^{(1)}))$ , or on  $d_{\text{Eden}}^\dagger(f_n, B_{\lfloor \alpha_j n^{1/4} \rfloor}^\bullet(\mathcal{T}_n^{(1)}))$ .

Once (74) has been established, we obtain the statement of Theorem 3 via a straightforward adaptation of the proof of Theorem 1. Lemma 28 is used once again to verify that if we pick independently a sufficiently large number  $N$  of vertices uniformly distributed over  $\mathcal{V}(\mathcal{T}_n)$ , then, with high probability uniformly in  $n$ , any face will be within dual (or Eden) distance at most  $\varepsilon n^{1/4}$  of one of the faces incident to these vertices.  $\square$

*Proof of Theorem 4.* This proof goes through by exactly the same absolute continuity argument as in the proof of Theorem 2, modulo of course the replacement of  $\mathbf{c}_0$  by  $\mathbf{c}_1$  or  $\mathbf{c}_2$ .  $\square$

## Appendix A1

In this appendix, we give a precise justification of (66). To this end, we need to obtain a refined version of the convergence of rescaled triangulations to the Brownian map [27]. We will verify that this convergence holds in the sense of the Gromov–Hausdorff–Prokhorov metric, if the vertex set of the triangulations is equipped with the uniform probability measure, and the Brownian map with its canonical volume measure. We refer the reader to [32, Section 6.2] for the definition of the Gromov–Hausdorff–Prokhorov metric.

**Theorem 6.** *Let  $\mathcal{T}_n$  be a uniformly distributed rooted plane triangulation with  $n + 1$  vertices, and let  $d_{\text{gr}}^n$  denote the graph distance on  $\mathcal{V}(\mathcal{T}_n)$ . If  $\mu_n$  denotes the uniform probability measure on  $\mathcal{V}(\mathcal{T}_n)$ , we have*

$$(\mathcal{V}(\mathcal{T}_n), 3^{1/4} n^{-1/4} d_{\text{gr}}^n, \mu_n) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{m}_\infty, D^*, \mu),$$

*in distribution for the Gromov–Hausdorff–Prokhorov topology, where  $(\mathbf{m}_\infty, D^*)$  is the Brownian map and  $\mu$  is the volume measure on  $\mathbf{m}_\infty$ .*

*Proof.* We first recall that the Brownian map is defined in terms of a random continuous function  $D^*$  on  $[0, 1] \times [0, 1]$ . The mapping  $(s, t) \mapsto D^*(s, t)$  is a pseudo-metric on  $[0, 1]$ , and if one considers the associated equivalence relation (namely,  $s \sim t$  if and only if  $D^*(s, t) = 0$ ), the Brownian map  $\mathbf{m}_\infty$  is the quotient space  $[0, 1]/\sim$ , which is equipped with the distance induced by  $D^*$ . We write  $\mathbf{p}$  for the projection from  $[0, 1]$  onto  $\mathbf{m}_\infty$  and note that  $\mathbf{p}(1) = \mathbf{p}(0)$ . The volume measure  $\mu$  on  $\mathbf{m}_\infty$  is just the image of Lebesgue measure on  $[0, 1]$  under  $\mathbf{p}$ .

Let us recall some ingredients from the proof in [27, Section 8], to which we refer for more details. It is convenient to consider a vertex  $o_n$  uniformly distributed over  $\mathcal{V}(\mathcal{T}_n)$ . Note that it is enough to prove that the convergence of the theorem holds when  $\mathcal{V}(\mathcal{T}_n)$  is replaced by  $\mathcal{V}(\mathcal{T}_n) \setminus \{o_n\}$  and  $\mu_n$  is replaced by the uniform probability measure  $\nu_n$  on  $\mathcal{V}(\mathcal{T}_n) \setminus \{o_n\}$ .

Say that an edge of  $\mathcal{T}_n$  is special if both ends of this edge are at the same graph distance from  $o_n$  (in particular, loops are special). Define another planar map  $\tilde{\mathcal{T}}_n$  by adding a new vertex at the “middle”

of every special edge, and write  $V(\tilde{\mathcal{T}}_n) \supset V(\mathcal{T}_n)$  for the vertex set of  $\tilde{\mathcal{T}}_n$ . The function  $(u, v) \mapsto d_{\text{gr}}^n(u, v)$  defined on  $V(\mathcal{T}_n) \times V(\mathcal{T}_n)$  is then extended to  $V(\tilde{\mathcal{T}}_n) \times V(\tilde{\mathcal{T}}_n)$  by declaring that the distance  $d_{\text{gr}}^n(u, v)$  between  $u \in V(\tilde{\mathcal{T}}_n)$  and  $v \in V(\tilde{\mathcal{T}}_n)$  is the minimal length of a path from  $u$  to  $v$ , assuming that edges of  $\tilde{\mathcal{T}}_n$  that correspond to non-special edges of  $\mathcal{T}_n$  have length 1 (as usual) whereas edges of  $\tilde{\mathcal{T}}_n$  obtained by the splitting of a special edge of  $\mathcal{T}_n$  have length  $1/2$  (see [27, Section 8.3] for more details).

According to [27, Section 8], we can find an integer  $k_n \geq n$  (which depends on  $\mathcal{T}_n$ ) and a mapping  $j \mapsto v_j^n$  from  $\{0, 1, 2, \dots, k_n\}$  onto  $V(\tilde{\mathcal{T}}_n) \setminus \{o_n\}$  (called the white contour sequence in [27]), such that we have the convergence in distribution

$$\left(3^{1/4}n^{-1/4}d_{\text{gr}}^n(v_{\lfloor k_n s \rfloor}^n, v_{\lfloor k_n t \rfloor}^n)\right)_{s,t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} \left(D^*(s, t)\right)_{s,t \in [0,1]} \quad (75)$$

in the sense of the uniform convergence of continuous functions on  $[0, 1]^2$ . We refer to (58) and (59) in [27] for the convergence (75), which is indeed a key ingredient of the proof of the convergence of rescaled triangulations to the Brownian map. By using the Skorokhod representation theorem, we may and will assume that the triangulations  $\mathcal{T}_n$  have been constructed so that the convergence (75) holds a.s.

Write  $w_0^n, w_1^n, \dots, w_{n-1}^n$  for the vertices of  $V(\mathcal{T}_n) \setminus \{o_n\}$  listed in their order of appearance in the sequence  $v_0^n, v_1^n, \dots, v_{k_n}^n$ . For every  $j \in \{0, 1, \dots, k_n\}$ , let  $L_j^n$  be the number of distinct vertices of  $V(\mathcal{T}_n) \setminus \{o_n\}$  in the sequence  $v_0^n, \dots, v_j^n$ . By Proposition 8.2 in [27], we have

$$\sup_{0 \leq t \leq 1} |n^{-1}L_{\lfloor k_n t \rfloor}^n - t| \xrightarrow[n \rightarrow \infty]{} 0, \quad (76)$$

in probability. Also set  $\Lambda_i^n = \min\{j \in \{0, 1, \dots, k_n\} : v_j^n = w_i^n\}$ , for every  $i \in \{0, 1, \dots, n-1\}$ . As a straightforward consequence of (76), we have

$$\sup_{0 \leq t \leq 1} |k_n^{-1}\Lambda_{\lfloor nt \rfloor}^n - t| \xrightarrow[n \rightarrow \infty]{} 0, \quad (77)$$

in probability. Writing  $w_{\lfloor ns \rfloor}^n = v_{\Lambda_{\lfloor ns \rfloor}^n}^n$ , we can now combine the convergence (75) (assumed to hold a.s.) with (77) to get

$$\sup_{s,t \in [0,1]} \left| D^*(s, t) - 3^{1/4}n^{-1/4}d_{\text{gr}}^n(w_{\lfloor ns \rfloor}^n, w_{\lfloor nt \rfloor}^n) \right| \xrightarrow[n \rightarrow \infty]{} 0, \quad (78)$$

in probability.

Let  $\mathcal{R}_n = \{(\mathbf{p}(s), w_{\lfloor ns \rfloor}^n) : s \in [0, 1]\}$ , which is a compact correspondence between  $(\mathbf{m}_\infty, D^*)$  and  $(V(\mathcal{T}_n) \setminus \{o_n\}, d_{\text{gr}}^n)$ . Also let  $\pi$  be the probability measure on  $\mathbf{m}_\infty \times (V(\mathcal{T}_n) \setminus \{o_n\})$  defined by

$$\langle \pi, \phi \rangle = \int_0^1 ds \phi(\mathbf{p}(s), w_{\lfloor ns \rfloor}^n).$$

Plainly the first and second marginals of  $\pi$  are  $\mu$  and  $\nu_n$  respectively, and moreover  $\pi$  is supported on  $\mathcal{R}_n$  by construction. According to [32, Proposition 6], the desired Gromov–Hausdorff–Prokhorov convergence will follow if we can check that the distortion of  $\mathcal{R}_n$  converges to 0 in probability as  $n \rightarrow \infty$ . However, this is an immediate consequence of (78).  $\square$

Let us now explain why (66) follows from Theorem 6. As in (66), we consider, for every  $n \geq 1$ , a sequence  $(o_j^n)_{j \geq 1}$  of vertices chosen independently uniformly over  $V(\mathcal{T}_n)$ . From [32, Proposition 10] and the preceding theorem, we get that, for every  $k \geq 1$ , the random  $k$ -pointed metric spaces  $((V(\mathcal{T}_n), 3^{1/4}n^{-1/4}d_{\text{gr}}^n), (o_i^n)_{1 \leq i \leq k})$  converge in distribution, in the sense of the  $k$ -pointed Gromov–Hausdorff metric, to  $((\mathbf{m}_\infty, D^*), (\mathbf{p}(\xi_i))_{1 \leq i \leq k})$ , where  $\xi_1, \xi_2, \dots$  are i.i.d. uniform random variables on  $[0, 1]$ . On the other hand, for any fixed  $\delta > 0$  and  $\varepsilon > 0$ , we can choose an integer  $N$  such that, with probability

at least  $1 - \delta/2$ , any point of  $\mathbf{m}_\infty$  lies within distance at most  $\varepsilon/2$  from one of the points  $\mathbf{p}(\xi_1), \dots, \mathbf{p}(\xi_N)$ . Using the preceding convergence of random  $k$ -pointed metric spaces (with  $k = N$ ), we obtain that (66) holds for all sufficiently large  $n$ . The small values of  $n$  can then be handled by taking  $N$  even larger if necessary.

## Appendix A2

This appendix gathers a few estimates about vertex degrees in random triangulations. Although these results will not be surprising to experts of the field, we were not able to locate precise references dealing explicitly with our case of type I triangulations.

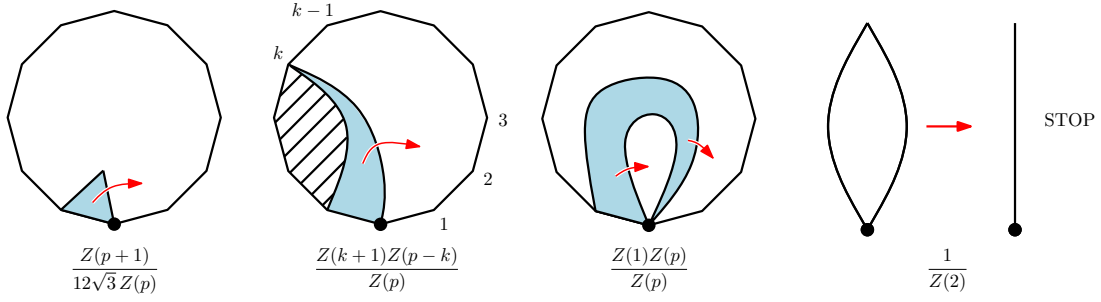
**Proposition 30.** *Let  $p \geq 1$  and let  $\mathcal{T}^{(p)}$  be a Boltzmann triangulation of the  $p$ -gon. We denote by  $\mathcal{D}_p$  the degree (i.e. the number of incident half-edges) of the root vertex in  $\mathcal{T}^{(p)}$ . There exist two constants  $K_0$  and  $\lambda < 1$  which do not depend on  $p$ , such that, for every  $k \geq 1$ ,*

$$\mathbb{P}(\mathcal{D}_p \geq k) \leq K_0 \lambda^k.$$

*Proof.* We denote the origin vertex of  $\mathcal{T}^{(p)}$  by  $\rho^{(p)}$ . The idea is to explore the neighborhood of  $\rho^{(p)}$  from left to right using the peeling process and discarding the parts that are useless to determine the degree of the root vertex. This is similar to the proof [8, Lemma 4.2] in the case of type II triangulations, but the case of type I triangulations is trickier. The peeling process of Boltzmann triangulations is studied in detail in [11], and we will briefly recall the properties that we need.

We assume that  $\mathcal{T}^{(p)}$  is drawn in the plane so that the unbounded face is the bottom face, and the bottom cycle is then oriented counterclockwise (in agreement with our convention that the bottom face lies on the right of the root edge). We start by revealing the (finite) face incident to the edge of the boundary whose terminal vertex is  $\rho^{(p)}$ . There are several possibilities, which are illustrated in Fig. 12 and whose respective probabilities are expressed in terms of the quantities  $Z(k)$  given in (8).

- The revealed triangle has a new vertex in  $\mathcal{T}^{(p)}$ . This event happens with a probability equal to  $\frac{Z(p+1)}{12\sqrt{3}Z(p)}$ . In this case, the remaining triangulation, obtained after removing the discovered triangle, is distributed as  $\mathcal{T}^{(p+1)}$ .
- The third vertex of the revealed triangle belongs to the boundary of  $\mathcal{T}^{(p)}$  and there are  $k$  edges of the boundary, for some  $k \in \{1, 2, \dots, p-1\}$ , on the path going from the root vertex to this third vertex along the boundary, in counterclockwise order. This event happens with probability  $\frac{Z(k+1)Z(p-k)}{Z(p)}$ . On this event, the removal of the discovered triangle splits the triangulation into two subtriangulations which are distributed respectively as  $\mathcal{T}^{(k+1)}$  and  $\mathcal{T}^{(p-k)}$ . For the remaining part of the argument, we need only consider the subtriangulation (distributed as  $\mathcal{T}^{(k+1)}$ ) whose boundary contains the root vertex (we discard the hatched part in Fig. 12).
- The third vertex of the revealed triangle is the root vertex  $\rho^{(p)}$ . On this event, which happens with probability  $Z(1)$ , the root vertex is incident to two subtriangulations distributed respectively as  $\mathcal{T}^{(1)}$  and  $\mathcal{T}^{(p)}$ . We then need to continue the exploration in each of these subtriangulations (we may say that the exploration branches).
- Finally, when  $p = 2$ , there is a special case: with probability  $Z(2)^{-1}$  the triangulation of the 2-gon that we obtain is the edge-triangulation and the exploration process stops.



**Figure 12:** Discovering the triangle incident to the edge of the boundary whose terminal vertex is the root vertex. In the first two cases on the left, we continue discovering the triangles incident to the root vertex in the “unknown” part of the triangulation that is incident to the root vertex. In the third case, we need to continue the exploration in the two unknown parts (they are both incident to the root vertex). In the last case the exploration stops.

The above exploration allows us to discover the degree of the root vertex. Note that the exploration branches when the peeling of a face splits the triangulation into two subtriangulations that are both incident to the root vertex. Recording the perimeters of the successive “subtriangulations” incident to the root vertex that pop up during the exploration leads to a (discrete time) multi-type branching process  $\mathcal{B}$  where the types of the particles are in  $\{1, 2, 3, \dots\}$ . Furthermore, the branching transitions are described as follows: for  $p \geq 1$  and  $1 \leq k \leq p-1$ ,

an individual of type $p$ has 1 child of type $p+1$	with probab. $\frac{Z(p+1)}{12\sqrt{3}Z(p)}$
an individual of type $p$ has 1 child of type $k+1$	with probab. $\frac{Z(k+1)Z(p-k)}{Z(p)}$
an individual of type $p$ has 2 children of respective types 1 and $p$	with probab. $Z(1)$
an individual of type 2 has no child	with probab. $Z(2)^{-1}$ .

(79)

The total degree of the root vertex in  $\mathcal{T}^{(p)}$  is bounded above by twice the total number of individuals in the branching process  $\mathcal{B}$  starting from a single particle of type  $p \geq 1$ . The proof of the proposition is then completed by the following lemma.

**Lemma 31.** *Let  $\mathcal{N}_p$  be the total number of particles in a multi-type branching process with branching transitions described in (79) and started from a single particle of type  $p \geq 1$ . Then there exist two positive constants  $K_1 > 0$  and  $0 < \tilde{\lambda} < 1$ , which do not depend on  $p$ , such that, for all  $k \geq 1$ ,*

$$\mathbb{P}(\mathcal{N}_p \geq k) \leq K_1 \tilde{\lambda}^k.$$

*Proof.* We define another multi-type branching process  $\mathcal{B}'$  with only 3 types of particles called **1**, **2** and  $\bar{\mathbf{3}}$ , whose branching transitions are described as follows:

an individual of type <b>1</b> has 1 child of type <b>2</b>	with probab. $\frac{Z(2)}{12\sqrt{3}Z(1)} = 1 - Z(1)$
an individual of type <b>2</b> has 1 child of type $\bar{\mathbf{3}}$	with probab. $\frac{Z(3)}{12\sqrt{3}Z(2)}$
an individual of type $\bar{\mathbf{3}}$ has 1 child of type $\bar{\mathbf{3}}$	with probab. $1 - Z(1) - \frac{Z(2)}{12}$
an individual of type <b>2</b> has 1 child of type <b>2</b>	with probab. $Z(1)$
an individual of type $\bar{\mathbf{3}}$ has 1 child of type <b>2</b>	with probab. $\frac{Z(2)}{12}$
an individual of type $\mathbf{p} \in \{\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}}\}$ has 2 children of types 1 and $p$	with probab. $Z(1)$
an individual of type <b>2</b> has no child	with probab. $Z(2)^{-1}$ .



We can interpret these transition probabilities as follows. Starting from either type **1** or type **2**, the transition probabilities are the same as in (79) except that all types  $p \geq 3$  are merged into a single type  $\bar{3}$ . The probability starting from type  $\bar{3}$  of having two children (of types **1** and  $\bar{3}$ ) is the same as the corresponding probability in (79). The other transitions from type  $\bar{3}$  are designed so that the following property holds. The probability of the transition  $\bar{3} \rightarrow \mathbf{2}$  in  $\mathcal{B}'$  is equal to  $Z(2)/12$  and is thus smaller, for any  $p \geq 3$ , than the probability of the transition  $p \rightarrow 2$  in  $\mathcal{B}$ , which is equal to

$$Z(2) \frac{Z(p-1)}{Z(p)} = Z(2) \frac{p}{6(2p-5)}.$$

Note that the probability of the last transition  $\bar{3} \rightarrow \bar{3}$  is set to  $1 - Z(1) - Z(2)/12$  so that the sum of the transitions starting from  $\bar{3}$  is equal to 1. Using the preceding remarks, it is then easy to verify that we can couple a branching process  $\mathcal{B}$  starting from a single particle  $p$  and a branching process  $\mathcal{B}'$  starting from a single particle of type **1** if  $p = 1$ , of type **2** if  $p = 2$ , and of type  $\bar{3}$  if  $p \geq 3$ , so that the total number of particles in  $\mathcal{B}'$  is larger than that in  $\mathcal{B}$ .

Now observe that the matrix of the mean offspring numbers of each type in  $\mathcal{B}'$  is given by

$$\begin{pmatrix} 2Z(1) & Z(1) & Z(1) \\ \frac{Z(2)}{12\sqrt{3}Z(1)} & 2Z(1) & \frac{Z(2)}{12} \\ 0 & \frac{Z(3)}{12\sqrt{3}Z(2)} & 1 - \frac{Z(2)}{12} \end{pmatrix}$$

and from the explicit formulas (8) one checks that the spectral radius of this matrix is  $0.917457... < 1$ . It follows by classical results (see [9, Chapter V]) that the total number of particles in  $\mathcal{B}$  (starting from any of the three possible types) has an exponential tail. This completes the proof of the proposition.  $\square$

Recall our notation  $\mathcal{T}_n$  for a uniformly distributed plane triangulation with  $n + 1$  vertices.

**Lemma 32.** *Let  $\text{MD}(\mathcal{T}_n)$  be the maximal degree of a vertex in  $\mathcal{T}_n$ . There exists  $A > 0$  such that*

$$\mathbb{P}(\text{MD}(\mathcal{T}_n) > A \log n) \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* We write  $\deg_G(x)$  for the degree of a vertex  $x$  in a graph  $G$  to avoid confusion. Let  $\mathcal{T}_n^{(1)}$  be a uniform triangulation of the 1-gon with  $n$  inner vertices. One can assume that  $\mathcal{T}_n$  is obtained from  $\mathcal{T}_n^{(1)}$  via the transformation of Fig. 2. In particular the root vertex  $\rho_n$  of  $\mathcal{T}_n$  is also the root vertex of  $\mathcal{T}_n^{(1)}$ , and we have

$$\deg_{\mathcal{T}_n}(\rho_n) \leq \deg_{\mathcal{T}_n^{(1)}}(\rho_n).$$

On the other hand, for any  $k \geq 1$ , if  $\mathcal{T}^{(1)}$  denotes a Boltzmann triangulation of the 1-gon and  $\rho^{(1)}$  stands for its root vertex,

$$\mathbb{P}(\deg_{\mathcal{T}_n^{(1)}}(\rho_n) \geq k) = \mathbb{P}(\deg_{\mathcal{T}^{(1)}}(\rho^{(1)}) \geq k \mid \#N(\mathcal{T}^{(1)}) = n) \leq \frac{Z(1)}{(12\sqrt{3})^{-n} \#T_{n,1}} \mathbb{P}(\deg_{\mathcal{T}^{(1)}}(\rho^{(1)}) \geq k).$$

Using (6) and the case  $p = 1$  of Proposition 30, we get for some constants  $C > 0$  and  $\lambda \in (0, 1)$ ,

$$\mathbb{P}(\deg_{\mathcal{T}_n}(\rho_n) \geq k) \leq \mathbb{P}(\deg_{\mathcal{T}_n^{(1)}}(\rho_n) \geq k) \leq Cn^{5/2}\lambda^k.$$

We finally use the same argument as in the proof of Theorem 1 and we get by re-rooting invariance that,



for every  $k \geq 1$ ,

$$\begin{aligned}
\mathbb{P}(\exists x \in \mathcal{V}(\mathcal{T}_n) : \deg_{\mathcal{T}_n}(x) \geq k) &\leq \mathbb{E} \left[ \sum_{x \in \mathcal{V}(\mathcal{T}_n)} \mathbf{1}_{\deg_{\mathcal{T}_n}(x) \geq k} \right] \\
&\leq \mathbb{E} \left[ \sum_{x \in \mathcal{V}(\mathcal{T}_n)} \deg_{\mathcal{T}_n}(x) \mathbf{1}_{\deg_{\mathcal{T}_n}(x) \geq k} \right] \\
&= 6(n-1) \mathbb{P}(\deg_{\mathcal{T}_n}(\rho_n) \geq k) \\
&\leq 6Cn^{7/2} \lambda^k.
\end{aligned}$$

Applying the last bound to  $k = A \log n$  with  $A > 4/|\log \lambda|$  yields the desired result.  $\square$

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